BUILDING BRIDGES TO ALGEBRA THROUGH A CONSTRUCTIONIST LEARNING ENVIRONMENT

Eirini Geraniou, Manolis Mavrikis
Institute of Education, London, UK

Abstract
In the digital era, it is imperative not to ignore the potential impact of digital technologies in education and how such tools can be successfully integrated in the mathematics classroom. We – similar to many others in the constructionism conference- have been inspired by the idea set out originally by Papert of providing students with appropriate vehicles for developing 'Mathematical Ways of Thinking'. A crucial issue though regarding 'vehicles' such as carefully designed digital tools is that of 'transfer' or 'bridging' i.e. what mathematical knowledge is transferred from students' interactions with such tools to other activities such as when they are doing 'paper-and-pencil' mathematics, undertaking traditional exam papers, or in other formal and informal settings. This paper will present some ideas formed based on data gathered as part of the MiGen project (www.migen.org) from studies aimed to investigate ways to build bridges to formal Algebra. Our more general aim was to support the implementation of digital technologies in the mathematics classroom, through carefully designed bridging activities that consolidate, support and sustain students' mathematical ways of thinking.

Keywords Algebraic generalisation and language, Transition, Exploratory learning, Bridging tasks

1. Background and Literature Review

Major transitions in a learner's life are much studied: the transition from counting to number [1], from number to arithmetic [2], from arithmetic to algebra [3]. The latter transition has been extensively researched and the mathematics education literature is replete with examples of student difficulties in learning algebra (e.g. [4]). Students struggle to understand the idea behind using letters to represent *any* value [5] and are inexperienced with using mathematical vocabulary to express generality. Even students capable of expressing a general rule through the use of words, like 'always' or 'every', struggle to use letters and symbols and form algebraic expressions. As [6] put it, students are routinely asked to "learn representation systems without anything to represent" (p. 546). Instead, the need to express and justify generality can be considered "the heart, root and purpose of algebra" ([7], p.2).

In the digital era, where digital technologies are increasingly making their appearance in the mathematics classroom students are faced with another transition, that of moving back and forth from paper-and-pencil to interacting with digital tools. It is imperative, therefore, to investigate how and whether students 'transfer' their knowledge from their interactions with digital tools to paper-and-pencil activities. We put transfer in quotes because it refers to different constructs for different people. There is of course a lot of research on 'transfer'. Our view is aligned with [8] who has argued that the metaphor should be viewed as transition instead of transfer, as crossing boundaries from one location to another is in fact a process of transition and he considers that people are the ones who move and not knowledge or learning. Other authors discuss about transfer in terms of the knowledge, which is transferred. Others claim that transfer "entails re-use of knowledge, demonstrated and/or acquired in one situation (or class of situations), in a "new" situation (or class of situations)" [9] . Similarly, [10] claims "Transfer of learning is our use of past learning when learning something new and the application of that learning to both similar and new situations. […]. Transfer of learning […] is the very foundation of learning, thinking and problem solving" (p.xiii).

When considering the 'transfer' of knowledge, one needs to consider what type of knowledge is being transferred, as the educational literature has revealed many different types of knowledge ([11]; [9]). For example, [12] used the existence or lack of connections between internal networks (schemas) to introduce conceptual and procedural knowledge. Conceptual knowledge consists of a connection of networks and is rich in relationships. Procedural knowledge is defined as the learning of a series of actions, where the only apparent connections are those between successive actions in the procedure. The latter type of knowledge is the one that usually is observed with students who interact with digital tools, as they discover how the tool 'works' and can rely on the immediate feedback they get from the tool but not necessarily reflect upon their actions nor verify their answers [13]. In the case of Logo, for example, [13] had observed that students often ended up using non-mathematical strategies over mathematical strategies to determine turtle turns or segment sizes. Moreover, students tend to ignore interesting perspectives on mathematics while interacting and gaining more and more experience with digital tools (e.g. [14]). Saying that though, it is worth revisiting the arguments in [15] that students who interact with Logo can visit mathematically rich areas, which they would not have approached otherwise. Using digital tools which are specially designed to support students' difficulties and possible misconceptions on the topic of algebra for example, should 'smoothen' the transition to formal algebra without rendering it impossible for them to reach the mathematical bank of algebra. Besides 'learning' the tool and be experts in using it, students should then be able to make the connections to the maths. The issue is to find out ways for supporting students to make such connections.

In the case of Logo, [14] considered "the type of connections generally expected, and very seldom observed, between Logo practice and mathematics" (p.247) as *transfer* and suggested that "a rather long period of Logo practice (one that is rich in reflection) is necessary before transfer to mathematics can occur ([16])". He also used the 'bridging' metaphor to describe the connections students or educators try to build between different domains or topics within the same domain or aspects of the school life and the everyday life. We valued and aligned our work with the bridge metaphor (as opposed to the notion of transfer) as it allows connections to be identified and made as early as possible between the domain of the digital tool and mathematics and hopefully have a greater impact in students' learning.

Relevant research (e.g. [14]) and our anecdotal observations suggest that students rarely use ideas, concepts or strategies they seem to have acquired through their interactions with digital technologies in their mathematics classrooms. Using the Logo environment as an example, [14] has claimed that the tool's features which are designed to support students when faced with complex mathematical problems may impede them from making connections between their work in Logo and any mathematical or geometrical ideas they are already familiar with and use when problems seem less complex. A second reason [14] presented was the lack of information on why and how to build bridges to formal maths, which were not often made in standard Logo situations. He suggested that bridges could be built through presenting structured tasks, using appropriate microworlds and making explicit interventions during students' interactions. Another suggestion was to "instil in them an attitude of searching continually for possible connections" (p.255).

Considering all the issues discussed above, the research carried out in the MiGen[1] project has offered major gains in the understanding of students' development of algebraic ways of thinking through their interaction with exploratory learning environments. In this paper, we will present data gathered from 11-14 year old students who worked on bridging activities carefully designed to support their transition from interacting with the MiGen digital tool, namely eXpresser, to traditional paper-and-pencil algebraic generalisation tasks. General concluding remarks will be discussed on this transition to paper-and-pencil tasks, but to algebra in general too, and some ideas will be shared regarding the successful integration of the eXpresser tool, but also the successful integration of similar digital tools, into the mathematics classroom.

2. Methodology

Over the past 7 years, we have carried out a number of studies in 6 different schools in London, worked with 11 mathematics teachers and collected data from 553 students aged 11-14 years old. Our data comprises one-to-one and small groups of students' interviews and transcripts, video (mostly screen recordings) and audio files from interviews, one-to-one, small groups and classroom observations, detailed logs from students' interactions in the form of a database and interviews of teachers and transcripts and bridging activities specially designed to gauge students' knowledge.

We have presented results from our data analysis of our various studies in a number of papers (e.g. [17]; [18]; [19]; [20]). In this paper, however, we focus on the data collected from the bridging activities students worked on during, but mostly after their final interaction with the eXpresser tool, and we present our initial analysis. Since the eXpresser tool has been specially designed to support students dealing with some well-known and researched misconceptions on Algebra ([18]), the goal of this preliminary analysis was to identify the impact of those design decisions on students' understanding and reasoning about algebraic generalisation, whether students use any of the strategies they were 'trained' to use while interacting with eXpresser on the paper-and-pencil tasks and are successful in solving figural generalisation tasks. In particular, we focused on two algebraic ways of thinking, as we had described them in our previous work ([17]): (i) *Perceiving structure and exploiting its power*, which is about noticing what stays the same and what is repeated in a figural sequence so as to understand how the sequence is 'structured', supporting therefore "the development of structural reasoning and the habits of "breaking things into parts" by identifying "the building blocks of a structure" ([20])" ([17], p.69); and (ii) *Recognising and articulating generalisations, including expressing them symbolically*, which is the process of translating the observed structure in an algebraic expression, using formal algebraic notation to write general rules for numerical sequences. Students' answers were viewed several times and are being analysed in the light of these two algebraic ways of thinking.

In the results section, we share some initial insights gained from this data on bridging activities and students' strategies to solve figural pattern generalisation patterns without the support or immediate feedback of the eXpresser tool.

[1] The MiGen project was funded by the ESRC/EPSRC Teaching and Learning Research Programme (Award no: RES-139-25-0381) and the MiGen Follow-on project is funded by ESRC (Award no: ES/J02077X/1)

3. The eXpresser tool and Bridging Activities

The MiGen system is a pedagogical and technical environment that improves 11-14 year-old students' learning of algebraic generalisation, a specific and fundamental mathematical way of thinking. Its core component consists of a microworld, eXpresser, which supports students in their reasoning and problem-solving of a class of generalisation tasks. In eXpresser, students construct figural patterns by expressing their structure through repeated building blocks of square tiles, and articulating the rules that underpin the calculation of the number of tiles in the patterns. A typical activity in eXpresser will ask the student to reproduce a dynamic model (or part of it) presented in a window that appears on the side of the activity screen.

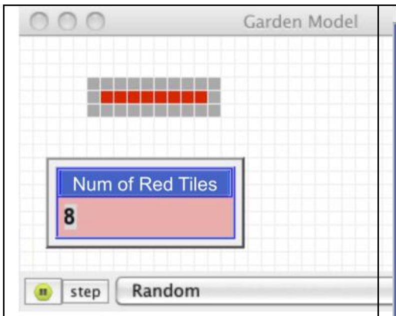


Figure 1. *A model for 8 red tiles surrounded by grey tiles. Students must construct a general model and find the general rule*

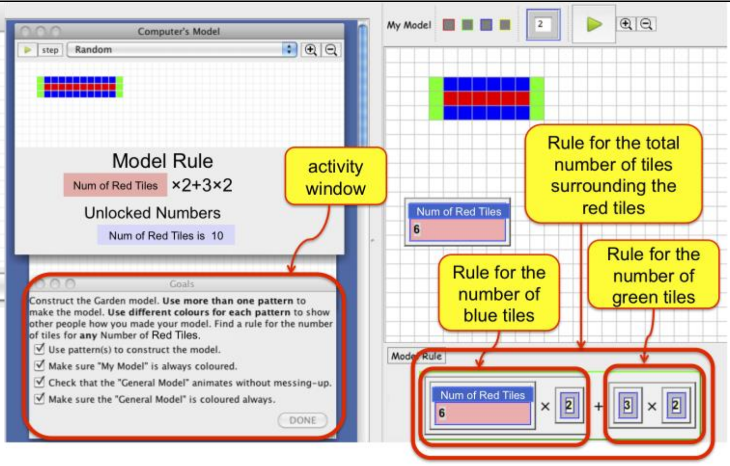


Figure 2. *The eXpresser screen showing the general and specific models (Computer's Model on the left, and My Model on the right), and a correct rule for the total number of surrounding tiles. The system has recognised that all the goals shown in the "Activity window" (lower left-hand corner) have been accomplished.*

Figure 1 shows a model where a row of red tiles is surrounded by grey tiles. Students are asked to construct a model that works for *any number* of red tiles, and find a rule for the total number of tiles surrounding the red tiles. They can test generality by *animating the model:* that is, by letting the computer change the number of red tiles at random. The design of eXpresser capitalises on animated feedback and on the simultaneous representation of a *specific* and *general* model ('My Model' and 'Computer's Model' in Figure 2), built by combining patterns and on the close alignment of the symbolic expression, the *Model Rule* and the structure of the model. In the *Computer's Model*, a value of the variable[2] ('Num of Red Tiles' in this example) is chosen automatically at random (it is '10' in Figure 2) which will generally be different from that in the specific model ('6' in Figure 2). So the *Computer's Model* indicates to students whether their constructions are *structurally* correct for the different values of the variable(s) assigned to the various properties. Students also construct a *model rule* for the total number of tiles, and validation of its correctness is made evident by colouring: tilings are *only* coloured if the rule for the number required is correct.

[2] All numbers in eXpresser are *constants* by default, referred to as 'locked' numbers. When the user 'unlocks a number', it is possible to change its value; it becomes a *variable*.

In our studies, students are presented with a sequence of activities. Initially, they are familiarised with eXpresser in two lessons through a number of introductory tasks and practice tasks, asking students to construct figural models. Afterwards they are given individual activities, such as the one described above (see figure 1). Students are asked to construct the task model in eXpresser using different patterns and combinations of patterns, depending on their perceptions of the task model's structure and derive a general rule for the number of square tiles needed for any Model Number. There are progressively harder tasks students can work on in eXpresser[3]. In our initial studies, students were presented with off-computer tasks, immediately after the final eXpresser task in an effort to reveal their strategies on solving similar tasks on paper and whether eXpresser had an impact on those strategies or not. In later studies, though, and after close collaboration with teachers, we recognised the need of activities, which promote students' reflections upon mathematical concepts and problem-solving strategies they used *throughout* their interactions with eXpresser and not just at the end. These we referred to as consolidation tasks and were used with 175 students out of the total 553 students we have worked with. So, throughout their interactions with eXpresser and immediately afterwards, students are presented with a number of bridging activities, which are designed to support their transition to paper-and-pencil tasks. We have designed 4 types of bridging activities: (i) Consolidation tasks, which are usually short tasks that are used to intervene and encourage students to reflect on their interactions with eXpresser throughout a sequence of eXpresser tasks, (ii) Collaborative tasks, which are presented at the end of an eXpresser task and focus on students' justification strategies regarding the equivalence or non-equivalence of their derived rules, (iii) eXpresser-like paper tasks, which are figural pattern generalisation tasks that are presented on paper, and (iv) text-book or exam like tasks, which are the traditional generalisation tasks given to students on paper. These are presented in Figure 3.

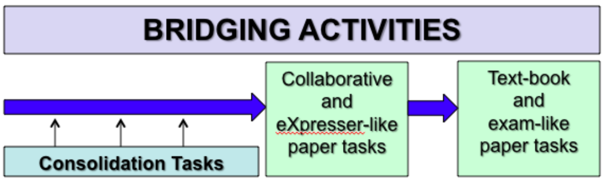
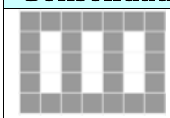
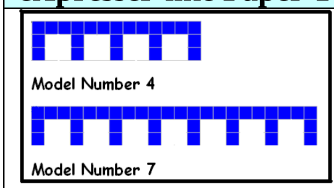


Figure 3. *The schematic presentation of the Bridging activities.*

For the purposes of this paper, we will only focus on all types of bridging activities, except for the collaborative ones[4]. Examples of the 3 types of bridging activities are presented in Figure 4.

Consolidation Task – *Train-track*	
	1. How many tiles are needed for Models 4, 8,1 and 100? 2. If we use 'M' to stand for the model number, how many tiles are needed for Model M? 3. Use the space below to explain the different parts of your rule – use the diagrams left or your own if it helps.
eXpresser-like Paper Task – *Bridges*	
 Model Number 4 Model Number 7	1. Find the rule for the number of tiles for any Model Number. 2. Find the number of tiles for Model Number 5, 10 and 100.

[3] Some eXpresser tasks can be found on expresser.lkl.ac.uk

[4] Some initial results on the collaborative activities are presented in [19].

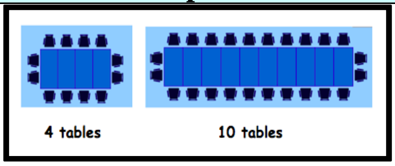
Text-book Paper Task – *Tables and Chairs*	
	1. Find the general rule for the number of chairs for any number of tables. 2. Use your rule to find the number of chairs for 20 tables and for 200 tables. 3. If I have 26 chairs, how many tables do I need?

Figure 4. *Examples of Bridging Activities.*

4. Results

Students were asked to work independently on the 3 types of tasks, examples of which are shown in Figure 4. Using the two algebraic ways of thinking, mentioned earlier, as an analytical framework for interpreting students' strategies when undertaking the bridging activities, we present our initial results under those two headings.

(i) Perceiving structure and exploiting its power

For the consolidation tasks, most of the 175 students demonstrated on the model figures presented on paper how they visualised the structure of the given model. In Figures 5, 6 and 7 below, we present some examples of students' answers on the 'Train-track' Consolidation task, the 'Bridges' eXpresser-like paper task and the 'Tables and Chairs' textbook-like task. They clearly marked the different parts that would remain the same in any instance of the pattern and the parts, which, repeated every time, create the different instances of the pattern. Some of them, perhaps influenced by the colouring feature of eXpresser, used coloured pens to identify these different building blocks. Students demonstrated a variety of ways to visualise the task patterns and it was evident how influenced they were by the eXpresser's features as they were using the eXpresser terminology, e.g. number of building blocks or models. For example, in Figures 6 [H], [I] and [J], students drew the 2 building blocks that they could use if they were solving this task in eXpresser, that of a column of 3 square tiles and that of an 'L'- shaped one of 5 tiles. In all these examples, it is evident in how many different ways students have visualised the task models. There were a few students, who managed to derive a correct general rule, but they did not demonstrate on paper the structure in which they possibly visualised the pattern. Such an example is given in figure 6 [F].

(ii) *Recognising and articulating generalisations, including expressing them symbolically*

Students seemed to rely on the structure of the given task model in order to articulate a general rule. Most of them provided clear explanations in order to justify their derived rules and share their solution and revealed some fluency in using the formal algebraic language. They identified what stayed the same and translated that into a constant in their rule. For example, in Figure 6 [J], the student had even annotated their rule (5xM)+3. They showed that the coefficient 5 is the number of building blocks in their 2nd building block, which, as they claimed, is repeated. The constant 3 is the number of tiles in their 1st building block, which is not repeated. Similarly, the student in Figure 6 [I] successfully identified 2 building blocks, which can produce the task model, and indicated which building block stays the same and which is repeated.

Students' answers revealed their ability to articulate general statements, such as "with every new model, another 7 is added and if there's 'M' amount of models, it should be (7xM)+5" (Figure 5 [A]) or "there is always 2 chairs to the ends of the single tables, then 2 chairs on the end of all tables put together" (Figure 7 [K]). But the crucial step was their ability to translate that generalisation in parallel to their visualised structures into general rules. Most students used the eXpresser language and terms such as 'model number' to represent the variable in their rule (e.g. "5xwhatever model number n is+3”, Figure 6 [F], as an intermediate step before expressing their derived rules in a formal algebraic expression (e.g. “(5xM)+3”, Figure 6 [J]). eXpresser seems to have played a crucial role in this outcome, as it encourages students to name their variables (‘unlocked’ numbers) based on what its various values represent and therefore allows students to give meaning to that variable. This step has eased students’ transition to the formal algebraic language and seems to have given meaning to the use of letters to represent ‘unknown’ values.

Some students evaluated their rules, by using specific values for their variable and used these examples to justify further their derived rules (for example Figures 6 [J] and 7 [K].

Even though the presentation of each of the bridging activities has been carefully designed to prevent students from looking for the term-to-term rule in a sequence, there were some cases of students, especially in the text-book like bridging activities, who reverted to their past experiences and worked out the answers for each consecutive term in a sequence. For example, in Figure 7 [N], the student calculates the number of chairs when having 1 table, then 2 tables, then 3 tables, etc. Despite, their focus on the term-to-term rule, they managed to spot the correct general rule and wrote “Chairs=tablesx2+4”. It would have been interesting to investigate if this student would have used the same strategy when faced with a more complex figural pattern generalisation task.

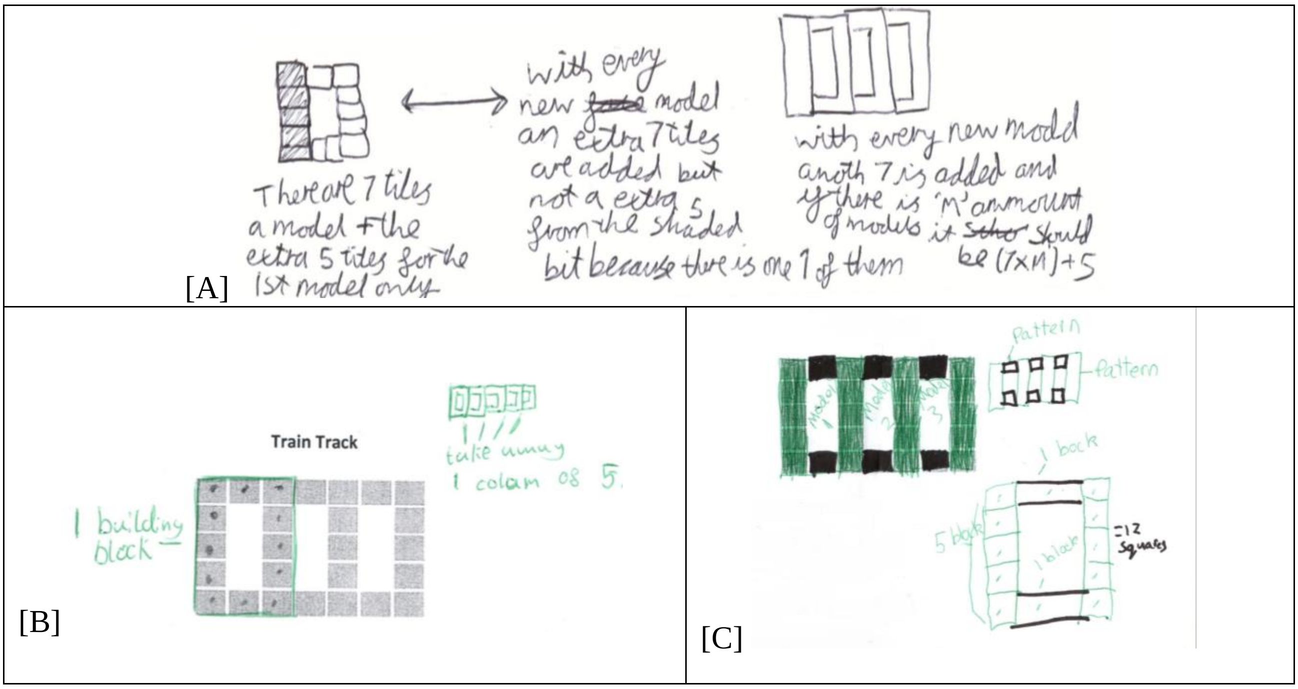


Figure 5. *Examples of students' answers on the Train-track consolidation bridging activity.*

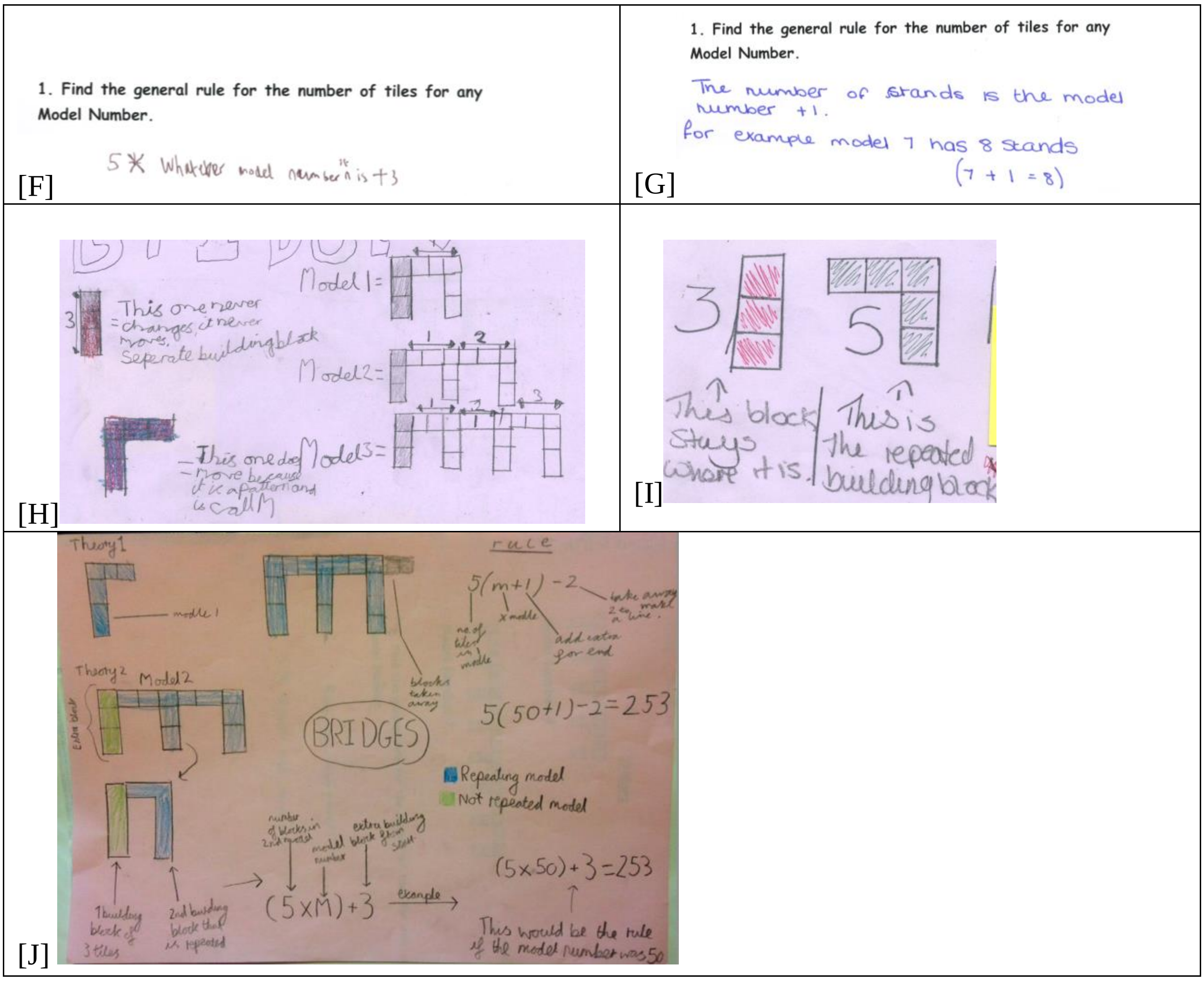


Figure 6. *Examples of students' work on the eXpresser-like Bridges bridging activity.*

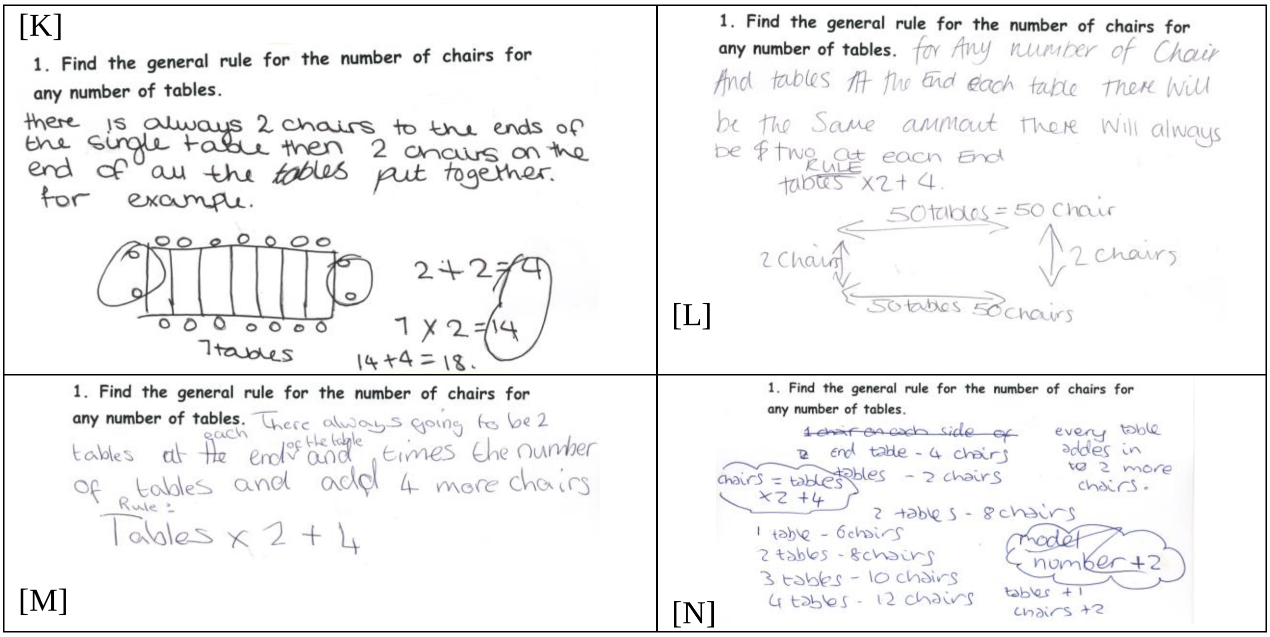


Figure 7. *Examples of students' work on the Tables and Chairs textbook-like bridging activity.*

5. Conclusion

Researcher: "so, what did you learn from interacting with the eXpresser tool?"
Student: "I learned how to put tiles and make nice patterns in different colours"
(a student's quote at an early pilot study with eXpresser)

Students nowadays are asked to interact with a number of ICT tools, which have been carefully designed and developed to engage them and support their learning of mathematics. Quite often though, students learn how to interact with the tool, create beautiful productions, such as colourful patterns as suggested by the student in the above quote, and often get to the right answer or 'an' answer, without necessary reflecting on and consolidating their knowledge during their interactions. They may know how to use the tool procedurally, but can fail in understanding conceptually the mathematical concepts and procedures that the tool was designed to help them with. Consequently, teachers can be hesitant in using such tools in their lessons as it is hard for them in their busy work lives to be convinced of the tools' short- and long-term value and can be reluctant to incorporate the use of ICT tools into mathematics instruction.

In the case of the eXpresser tool, and as it has hopefully been revealed in the few examples presented in the previous section, students seemed to have successfully 'transferred' their gained knowledge/understanding or crossed the 'bridge' from the eXpresser algebra to formal algebra. They have demonstrated their conceptual understanding of deriving a general rule and allowed us to claim they can generalise and adopt algebraic ways of thinking when solving paper and pencil figural pattern generalisation tasks. Our experience from the various studies for the MiGen project so far has supported the need for bridging activities, whose objective is to make the connections to algebra explicit. The need and value of such activities have been claimed by [16] too, who claimed that "the do-math-without-noticing-it philosophy of Logo can be abandoned in favour of techniques that *explicitly* present looking for connections between Logo and mathematics as an objective of a task" (p. 253). We also recognised, similarly to [16]'s research with Logo that "In contrast to the more classical transfer model, in which practice is usually necessary in one domain before any transfer can be obtained, useful bridges can be built from the beginning, as soon as work has started in both domains" (p. 265, [16]) and this for us was addressed by the consolidation tasks. There also seems to be the need of a long period of practice with the eXpresser tool, and any mathematics digital tool, rich in reflection and consolidation, before transfer to mathematics can be deemed possible. A view supported by other researchers in the past (e.g. [18], [16])

Finally, referring to the theme of Constructionism 2014 on creativity, we have seen eXpresser used in several creative ways from both students and teachers. Students have been given challenging tasks, such as to design eXpresser tasks to challenge their peers or create posters to share their views of eXpresser, its activities, what they believe they have learned during their interactions and identify similarities to traditional algebra. There are also several tools ranging from mathematical games to elaborate production or programming tools where students are given the opportunity to use or develop skills in order to solve puzzles or create and share artefacts, but the concern for us (and teachers we interact with daily) is the same as we mentioned above: What the residual knowledge that gets noticed by the interaction with such tool is and how we make it more explicit and support the learning of mathematics in constructionist learning environments.

A successful integration in our view involves the successful transition from interacting with a digital tool to the awareness of the knowledge that can potentially be transferred from students' interactions with digital technologies to paper-and-pencil activities and identifying the ways to encourage the sustainability of such knowledge. Taking into consideration this vision, our aim remains to investigate further the issues of 'Transfer' and 'Bridging' and support the implementation of digital tools in the classroom through carefully designed and innovative bridging activities that consolidate, support and sustain students' mathematical ways of thinking.

References

[1] Cobb, P. (1987). An analysis of three models of early number development. *Journal of Research in Mathematics Education*, 18, 163-179.

[2] Fuson, K. (1990). 'Conceptual structures and multi-digit numbers'. *Cognition and Instruction*, 7, pp. 343-404.

[3] Davis, R. B. (1985). Algebraic thinking in the early grades. *Journal of Mathematical Behavior*,4,195-208.

[4] Stacey, K. & Macgregor, M. (2002). Curriculum Reform and Approaches to Algebra. In Sutherland, R., Rojano, T., Bell, A. & Lins, R. (eds.), *Perspectives on School Algebra* (pp.141-153). The Netherlands: Springer.

[5] Duke, R. & Graham, A. (2007). Inside the letter. *Mathematics Teaching Incorporating Micromath, 200*, pp. 42-45.

[6] Kaput, J. J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515-556). New York: Macmillan.

[7] Mason, J. (2005) Frameworks for Learning, Teaching and Research: theory and practice, Plenary presentation, in S.Wilson et al.(Eds.), *Frameworks That Support Research & Learning*: Proceedings of the International Conference for the Psychology of Mathematics Education-NA.

[8] Beach, K. (2003) Consequential transitions: a developmental view of knowledge propagation through social organisations, in: T. Tuomi-Grohn & Y. Engestrom (Eds) *Between school and work: new perspectives on transfer and boundary-crossing* (Amsterdam, Elsevier Science).

[9] diSessa, A. A., & Wagner, J. F. (2005). What coordination has to say about transfer. In J. Mestre (Ed.), *Transfer of learning from a modern multi-disciplinary perspective* (pp. 121-154). Greenwich, CT: Information Age Publishing.

[10] Haskell, R.E. (2001). *Transfer of learning: cognition, instruction and reasoning*. Academic Press: San Diego.

[11] Beach, K. (1999) Consequential transitions: a socio-cultural expedition beyond transfer in education, *Review of Research in Education*, 28, 46–69.

[12] Hiebert, J. & Lefeivre, P. (1986). 'Conceptual and procedural knowledge in mathematics: an introductory analysis. In Hiebert (ed.) *Conceptual and Procedural Knowledge: The Case for Mathematics*, (pp. 1-27). Hillsdale, N.J.: Lawrence Erlbaum Associates.

[13] Hiller, J., Gurtner, J., and Kieran, C. (1988). Structuring and destructuring a solution: An example of problem-solving work with the computer. *Proceedings of the Twelfth International Conference fort he Psychology of Mathematics Education.* Budapest, July, 20-25, pp. 402-409.

[14] Gurtner, J. (1992). Between Logo and Mathematics: A Road of Tunnels and Bridges. In Hoyles, C. and Noss, R. (eds.), *Learning Mathematics and Logo*, The MIT press, Cambridge, Massachusetts, pp.247-268.
[15] Papert, S. (1980). *Mindstorms: Computers, Children and Powerful Ideas*. New York: Basic Books.
[16] Salomon, G., and Perkins, D. N. (1987). Transfer of cognitive skills from programming: When and how? *Journal of Educational Computing Research (3)*, pp. 149-169.
[17] Mavrikis, M., Noss, R., Hoyles, C. & Geraniou, E. (2013). Sowing the seeds of algebraic generalisation: designing epistemic affordances for an intelligent microworld. In Noss,R. & DiSessa,A. (eds.) Special Issue on Knowledge Transformation, Design and Technology, *Journal of Computer Assisted Learning*, Vol.29 (1), pp.68-84.
[18] Noss, R., Poulovassilis, A., Geraniou, E., Gutierrez-Santos, S., Hoyles, C., Kahn, K., Magoulas, G.D., and Mavrikis, M. (2012). The design of a system to support exploratory learning of algebraic generalisation. *Computers & Education*, 59(1), pp 63-81.
[19] Geraniou, E., Mavrikis, M., Hoyles, C. & Noss, R. (2011). Students' Justification Strategies on the Equivalence of Quasi-Algebraic Expressions. *Proceedings of the 35th International Conference for the Psychology of Mathematics Education,* 10^{th} – 15^{th} of July 2011, Ankara, Turkey.
[20] Cuoco, A., Goldenburg, E. & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curriculum. *Journal of Mathematical Behavior*, 15 (4), 375-402.