# *'How many circles are in the shape?'* Defining modelling-based Learning (MbL) through the iterative implementation of a specific activity to groups of early childhood teachers.

Chrystalla Papademetri-Kachrimani

European University Cyprus

***Abstract***

*In this paper we describe a modeling-based activity sequence which has been widely used by the author in teacher training courses with in service teachers and student teachers. Our aim is to describe how a continuous analysis of data which derived from the activity sequence implementation has allowed us (a) to define and better understand modeling-based learning and (b) provide scientifically justified teaching material for educating teachers about MbL via MbL, through an iterative procedure of desighning, implementing and re-designing the activity. Through the thorough description of the activity sequence and data collected from its implementation we conclude that MbL can be defined as a process where collecting data, observations and experiences in relation to a specific structure, phenomena or situation, the act of representation (constructing models) and problem solving interplay in a dynamic manner.*

***Keywords*** modelling-based learning, representations, teacher education

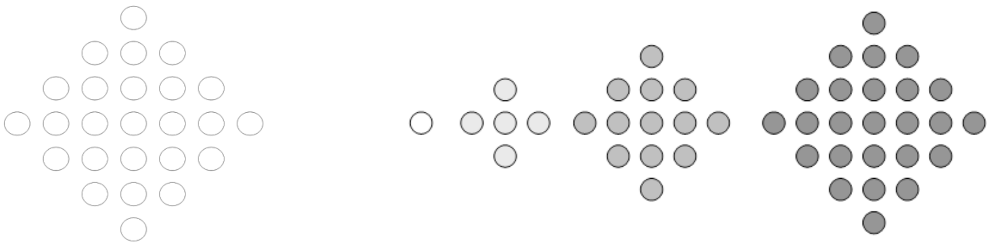
## 1. Introduction

Mathematical models are used to construct, interpret and mathematize real-world problems, or structures, phenomena and situations (English, Fox & Watters, 2005, English & Watters, 2004). Research has documented young children's ability for more complex and sophisticated forms of mathematical thinking and reasoning (Fox, 2006) and that young children 'can learn to model, generalize, and justify at earlier ages than traditionally believed possible, and that engaging in these practices provides students with early access to scientific and mathematical reasoning' (English & Watters, 2004, p.60). So, whereas there is a justification for the inclusion of modelling in education, MbL is not commonly incorporated into educational practice, especially in early ages. One obstacle for the incorporation of this approach into educational practices rises from the teacher's difficulty in understanding modeling-based learning and thus designing educational activities based on this approach. In this paper we describe a modeling-based activity sequence which is used in teacher training courses in order to allow teachers to (a) get a sense of how MbL feels like, (b) discuss and reflect upon the approach and define its main characteristics and consequently (c) be able to design and implement modeling-based learning in their practice.

The activity has been widely used by the author the last 5 years in postgraduate early childhood teacher training courses and professional courses for early childhood in service teachers. In all cases of implementation the instructor was regular at keeping fieldnotes which she used to reflect, revise and improve the activity sequence. From September 2009 till May 2011 the activity was used as a way to help teachers get involved in mathematical activity that would allow them to reflect and revise the way they conceived what ‘doing’ mathematics’ meant for them mainly through the way they experienced school mathematics as students and in contrast to how they conceived ‘doing mathematics’ in their classrooms with young children. From September 2011 the aim of using this activity sequence became more focused on educating teachers about MbL via MbL and has been tested through more systematic procedures of data collection (videotaped incidences of the teachers participating in the activity sequence and talking about modeling-based learning, teacher’s representations and the instructor’s fieldnotes) and analysis as part of a three year research project[1] involving planning, implementing, evaluating and scientifically justifying a joint mathematics and science literacy curriculum for early childhood education. The joint curriculum is developed by a mixed group of researchers, content-knowledge specialists and educators, based on a review of existing literature and applications in authentic early childhood settings. The research project involved two cycles of designing, implementing and reflecting upon mathematical activities by teachers as part of their involvement in long term professional development courses (duration 7 months). The research project led to the development of a process-based curriculum where learning (acquiring experiences, developing scientific skills, attitudes, conceptual understanding, and epistemological awareness) arises in a dynamic way through the involvement of children (3 to 6 year olds) in processes of problem solving, modeling and investigation. The activity sequence that is described in the following sections of this paper was implemented twice with two different groups of teachers as part of this research project and through its description and analysis we provide how it helps us better understand MbL and, at the same time, how it can be used to educate teachers about MbL.

## 2. *How many circles in the shape?* The Activity Sequence

The activity sequence[2] concerns the in depth study of the shape illustrated in Fig. 1a and the sequence which is created if we make the shape bigger or smaller (Fig. 1b). Table 1 provides an overview of the Activity Sequence (AS). The activities are described in more detail further on through a more thorough analysis of data collected from the implementations. It is important to stress out that during the implementation of the activities the teachers work in small groups.



(a) (b)

**Figure 1** The shape and sequence of shapes studied by the teachers

[1] The work reported in this paper was supported by a Cyprus Research Promotion Foundation Grand, #ΑΝΘΡΩΠΙΣΤΙΚΕΣ/ΠΑΙΔΙ/0609(ΒΕ)/14

[2] The idea for designing this activity sequence arose after participating in Discussion Group 9 (Mathematics and Kindergarten Teachers, A Challenge to the Research Community) coordinated by P. Tsamir, D. Tirosh, E. Levenson, M. Tabach και R. Barkai, at the 33rd Conference of the International Group for the Phychology of Mathematics Education, Thessaloniki, Greece, 19-24 July 2009.

| | Description |
|---|---|
| **Act1** | **How many circles in the shape?**<br>The teachers are asked to find as many different ways as they can to count the circles in the shape. They are given worksheets were the shape (Fig.1a) is printed many times. |
| **Act2** | **How many circles will there be if we make the shape bigger?**<br>The teachers are introduced with the sequence that is created by making the shape in Fig. 1a bigger and smaller. They are given the first four shapes of this sequence (Fig 2) and are asked to use solutions from the first activity to formulate a hypothesis in relation to how may circles will the fifth shape in the sequence have. The teachers are explicitly asked not to draw the fifth shape in order to answer the question. |
| **Act3** | **Programming a robot to make shapes with circles.**<br>The teachers are introduced with the idea of an imaginary robot that can follow very simple rules to construct shapes by using milk bottle lids. They are asked to formulate a set of very simple directions that when followed by the robot it will randomly end up constructing any of the shapes in the sequence. It is explained to the teachers that the robot (a) can only understand and follow very simple words and directions, (b) cannot count more than two objects and (c) does not understand any formal mathematical language |

**Table 1** Overview of the Activity Sequence (AS)[3]

# 3. Results from the teacher's involvement in the Activity Sequence (AS)

## 3.1. Results from the teacher's involvement in the first activity (Act1)

In Table 3 we see selected Solutions (S) from Act1 (Tab1). S1-14 are common solutions based on the frequency they appear among groups of teachers (they appear often by more than one teacher in all groups) whereas S15-28 appear more rarely (they only appear once in some groups). S29-32 have only appeared once in all the years the sequence is implemented.

The first important observation that one can make by looking at the solutions (Fig.2) is that they are represented in two different ways: through a graphical representation and an equation. An important remark needs to be done in relation to these two different representations. In the early circles of implementation these two representations were requested by the activity ('*Find different ways to count the circles in the shape and record them on the hand-out. Every time you find a different solution write the respective equation*'). After a few cycles of implementation and when the author started connecting the task sequence more explicitly with Mbl, these two representations were not requested by the activity. Consequently, these two different types of representation naturally arose through a need to represent a specific solution, thus making the modelling process more authentic. Thus in the more recent implementations the teachers use these two different types of representation in order to communicate their solutions. So they are encouraged to go backwards and forwards between what they actually saw in the shape and their representation(s), a process which results in a better understanding of their thinking and improvement of their representations. The need for the equation arises mostly through the need for a representation that gives a specific number answer to the original question (How many circles are there in the shape?). So whereas the graphical representation is very informative in relation to the way a teacher approached and solved the problem it does not give an answer to the original question. This is the case in all solutions besides S1 (Table 2). Overall, a close look at the solutions underpins the dynamic interplay between constructing an understanding (collecting observations) of the shape and representing this

[3] A similar task sequence concerning the shape and sequence in Fig.1 has been designed and implemented with groups of 4 to 6 olds as part of a Creative Learning and Play afternoon program concerning mathematics and science. The task sequence designed for children consists by the following three activities: (a) How can we make the shape prettier by using two different colors? (b) How many circles in the shape? (The children are asked to answer without having access to the shape), (c) How can I make the shape bigger/smaller? Which is the smallest/biggest shape I can make?

understanding. Thus, in order to address the original question the teachers start observing the shape and in order to remember, record and communicate the different solutions proceed with representations. At the same time the act of representing allows the teachers to better understand the shape (make more observations) and form strategies to find other, different solutions. But led us make this 'interplay' idea more clear through a closer look to the solutions represented in Table 2.

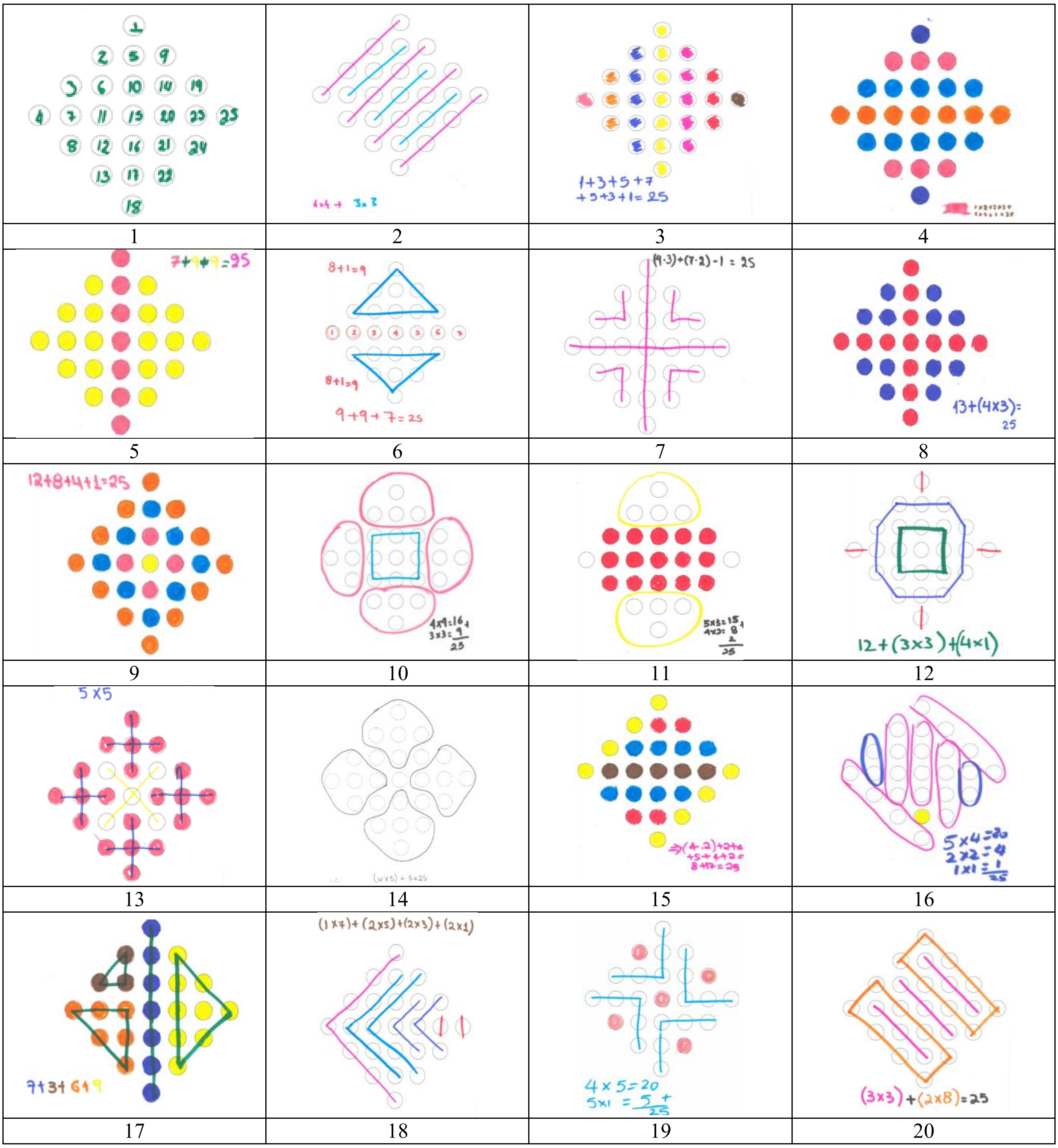



**Table 2** Selected solutions from the first activity of the sequence described in Table 1

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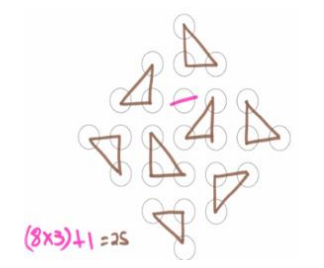
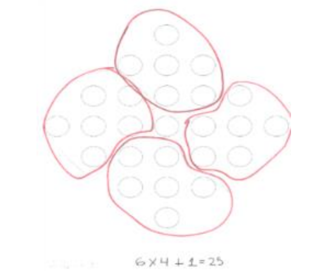
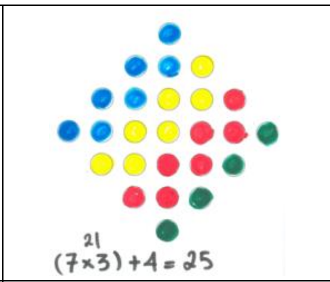
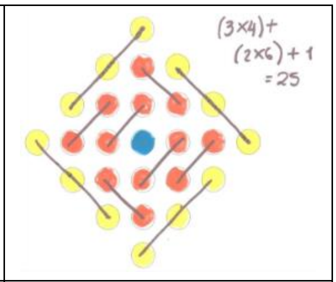
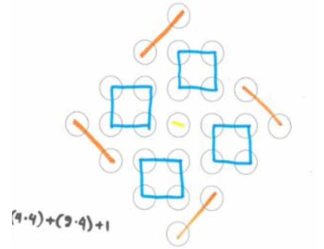
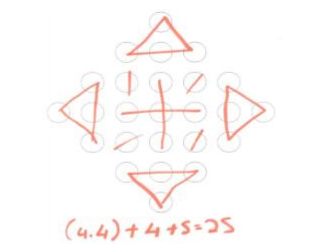
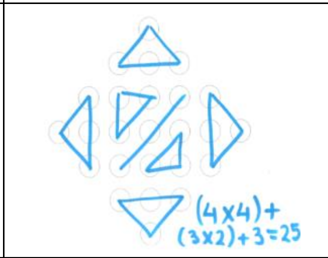
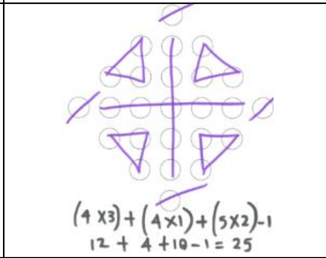
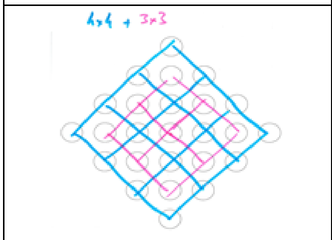
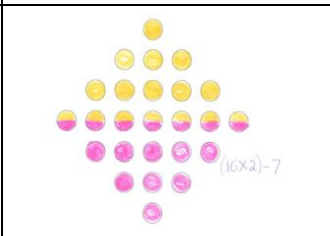
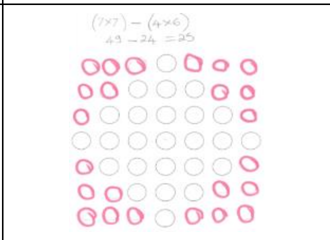
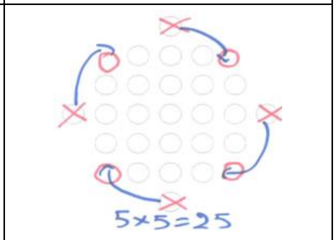
|---|---|---|---|
|   |   |   |   |
| 21 | 22 | 23 | 24 |
|   |   |   |   |
| 25 | 26 | 27 | 28 |
|   |   |   |   |
| 29 | 30 | 31 | 32 |

**Table 2** Selected solutions from Act1 (Table 1) (continued)

One important remark concerns the way in which the two types of representation (graphical and equation) can complement each other. The equation by its self cannot give a clear picture of a solution. This is obvious if we look at examples were the same equation corresponds to different graphical representations. For example equations 4X4+3X3=25, 5X5=25 and (2X1)+(2X3)+(2X5)+7=25 appear more than once but in many cases match to different solutions. See for example the following sets of solutions illustrated in Fig. 3: (2,10) (13,32) and (4,18) which correspond to the aforementioned equations. Even though the equation of each solution in each set is the same, the way the problem was approached was entirely different, and this is apparent by the graphical representation. Thus the graphical representation is an important source of information regarding the thinking and observation involved.

On the other hand, the graphical representation is not enough to fully comprehend a certain solution. Take for example S5-6 (Table 2). If we only had access to the graphical representations of these solutions we could easily be led to the conclusion that they actually represent the same solution. But the equation gives more information about the actual thinking involved and a better understanding of the graphical representation itself. So, whereas the teacher that came up with S5 'saw' two triangles consisting by 9 circles each, the teachers that came up with S6 'saw' two triangles consisting by a perimeter of 8 circles and a circle in the centre. The same stands for S7-8. Whereas the teacher that came up with S7 saw in the centre of the shape two lines consisting by 7 circles each, the teacher that came up with S8 saw a cross consisting by 13 circles.

So, in some solutions the equations help us to better understand the graphical representation. Take for instance S3-4 (Table 2). Whereas in both cases the teachers saw lines of circles the corresponding equation in each case is different: 1+3+5+7+5+3+1+25 (S3), 1X2+2X3+2X5+7=25 (S4). Thus in S4 the lines are grouped which is expressed in the graphical representation by the use of the same colour and translated in the equation by the use of multiplication. This proves that every small detail in a representation plays an important role and constitutes evidence of specific aspects of the way the shape was perceived. Look for example S2 and 29 that were found by the same teacher. They are so similar and yet so different. In S2 the shape is perceived as a shape consisting by lines of 4 and 3 circles forming a pattern whereas in S29 the shape was perceived as a shape consisting by two squares, a 4X4 and a 3X3 square. Another small detail is noteworthy in relation to these two solutions. The colours in the equations (blue and pink) correspond to the colours in the graphical representation. This shows a genuine effort to find ways in which the representations will have a communicative power meaning that they will have the power to describe the actual thinking involved without the need for further explanation.

In some cases by the teachers themselves and in some cases with the encouragement of the instructor the aim of the activity at some point focuses on 'How to count the circles in the shape by making the less possible counting?' As a result we have (a) strategies of grouping parts of the shape into lines and shapes (squares, triangles, rectangles) consisting by equal number of circles (S21-28, Table 2) (b) strategies of discovering rectangles within the shape which allows a quick way of counting circles (by multiplying the number of lines with the number of circles in each line) (S 10-12, Table 2) and (c) 'clever', creative solutions like the ones illustrated in S29-32 (Table 2). When the teacher that found S32 was asked how he thought about this solution he answered: '*I thought that since the answer to the problem is 25 we probably can find a solution by creating a 5X5 square. So I thought about moving around some of the circles to transform the shape into a 5X5 square.*' When the teacher that found S31 was asked how she thought about this solution she replied: *'We know that it is easy to find how many circles there are in a square. So I thought that if I add circles to create a square it will be easy to count how many circles there are in this square. And then all I have to do is to count how many circles I added in one corner and subtract that number by 4.*' The teacher that came up with S30 explained: '*The shape is symmetrical and we can cut it in two. Thus I only have to count half of the shape and then multiple that by two. But then I have to subtract the circles in the middle line because I counted them twice.'*

### 3.2. Results from the teacher's involvement in the second activity (Act2)

In Table 3 we have an overview of the different hypotheses (H) which have been formulated by teachers in regard to Act2 (Table 1). H1-3 are the most common ones and appear as alternative solutions by many teachers whereas H4-6 are more rare and have only appeared once. It is interesting to describe exactly what happens when this activity is presented to the teachers. So, the teachers in order to solve this problem and formulate hypotheses have in front of them the 4 first shapes of the sequence (Fig. 1b). Usually the first thing they do is that they ignore the activity directions, which ask them to *use solutions from the first activity* (Table1, Act2), and concentrate on observing the sequence. The first thing they normally notice is that each shape in the sequence is a repetition of the previous shape with the addition of a square of circles perimetrically. So they concentrate on the number of circles added each time which leads them to a sequence consisting with the multiples of 4 and thus hypothesize that the fifth shape in the row will have 16 circle more than the fourth (fig.2). When the teachers explain this hypothesis the instructor askes them to go back to the solutions from Act 1 and spot the solution that connects to this hypothesis (Table 3, 1). When they spot the solution the instructor asks the teachers to spot other solutions that might lead them to formulate different hypotheses that lead to the same result as before (the 5th shape will have 16 circles more than the 4th one). As a result the teachers end up with other hypotheses ( Table 3). So the teachers have in front of them solutions for Act 1 and each solution consists by a graphical representaion and an equation. It is noteworthy that in explaining their hypotheses to the instructor the teachers use the graphical representations rather than the patterns observed in the equations.

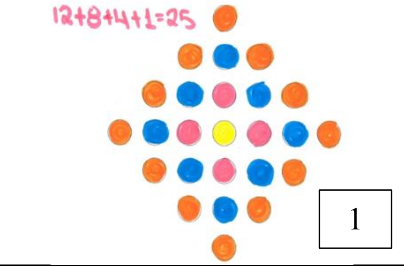
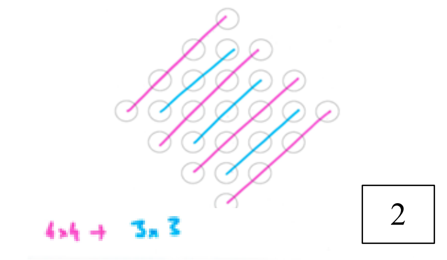
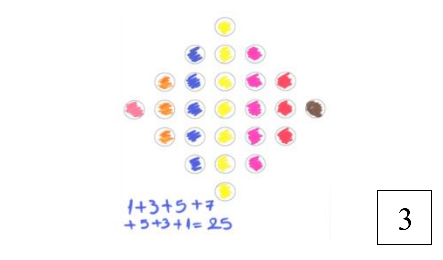
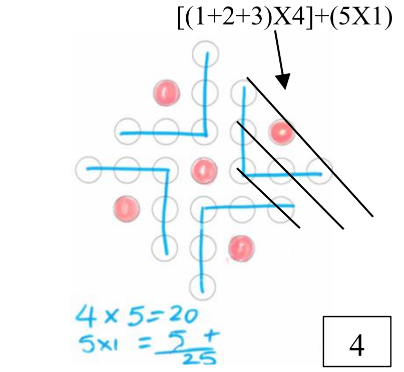
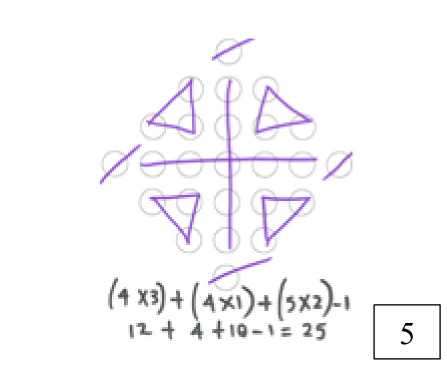
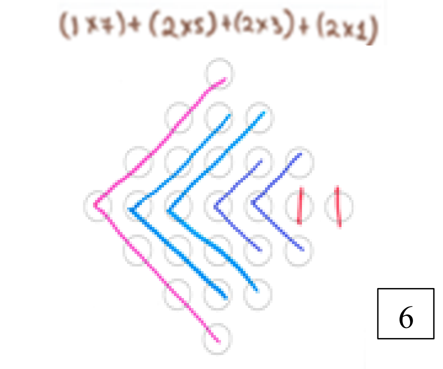
| | | |
|---|---|---|
|   |   |   |
| 16+12+8+4+1=41 | (5X5)+(4X4)=41 | 1+3+5+7+9+7+5+3+1=41 |
|   |   |   |
| [(1+2+3+4)X4]+(5X1)=41 | (4X6)+(4X1)+(7X2)-1=41 | (1X9)+(2X7)+(2X5)+(2X3)+(2X1)=41 |

**Table 3** Selected solutions from Act2 (Table 1)

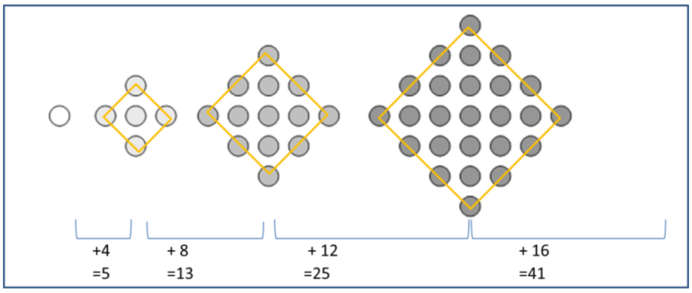



**Figure 2** Example of hypothesis formulated by the teachers based on S4 (Table2)

## 3.3. Results from the teacher's involvement in the third activity (Act3)

The many circles of implementations conducted over the years has allowed recording a number of possible behaviours that might occur from the teachers during their involvement in Act3 (Table 1) and create an instructors protocol (Table 4). Even though as an instructor I am always open to new solutions in this specific activity no group of teachers has managed to find a way to formulate rules that are based on a different way of seeing the shape besides S4 illustrated in Table 2. A set of rules which is based on this solution is the one described in Table 5. The instructors protocol (Table 4) is based on the set of rules described in Table 4. In the first row horizontally we have possible ‘problems’, difficulties’ or ‘mistakes’ that might arise in the course of the teacher’s involvement in the activity. In the second row we have ways in which the instructor may use these ‘mistakes’ in a constructive way. The number in the last row corresponds to the rules added to the set of rules (Table 5) as a result of each interaction. Even though the order in Table 4 is not arbitrary it does not mean that all groups of teachers go through all of these steps in the order presented here. Additionally the possibilities described in Table 4 do not cover all of the possible things that might arise during the course of the activity.

| The teachers often …. | | The instructor | Rule |
|---|---|---|---|
| 1 | … formulate rules that correspond to a specific shape in the sequence …. | … helps the teachers understand the idea of ‘randomness’.<br>… explains in an explicit manner that by following the rules the robot must be able to end up with any one of the shapes belonging to this ‘endless’ sequence.<br>…asks the teachers to imagine that the robot has a screen, they will ‘give’ their rules to the robot and the shape will appear on the screen. They must make sure that (a) the shape that will appear on the screen will be one of the shapes belonging to the sequence (b) that every time they ‘give’ the robot the same rules the robot may end up with a different shape from the sequence (c) that it must be possible for the robot to end up with any one of the shapes belonging to the sequence. | 1 |
| 2 | … base their rules on an understanding of the shape which does not allow them to formulate simple rules (e.g seeing the shape as a shape with a cross in the middle Fig. 2, S11) | … problematizes the teachers as to which of the solutions from the first activity is more simple and might allow us to formulate simple rules? (e.g. seeing the shape as a shape consisting by horizontal lines of circles, Fig. 2, S4)<br>… pretends to be the robot and follows the rules by interpreting the rules in a different manner than the one intended by the teachers. For example if the rules say take lids and make a cross the instructor makes a cross looking like this † and not like this +. After a number of obstacles the teachers wonder whether there original choice was not the most effective one and thus change their approach. | 3 |
| 3 | … formulate rules for constructing the shape from top to bottom or the other way round: from bottom to top. | … problematizes the teachers as to what should be the starting point? (e.g. beginning from the middle/larger horizontal line)<br>….. tries to connect this with the idea of random. ‘*So if we are to begin by asking the robot to pick up any number of lids so as to assure the aspect of randomness which part of the shape should the robot use these lids for?*’ | |
| 4 | … formulate rules for making a shape where the circles in each horizontal line are in between the circles of the line above it | …. pretends to be the robot and constructs the shape based on the rules. The shape looks different that the shape under study (‘thinner’). This makes the teachers to go back and look at the shape more carefully and compare it with the one that derived from their rules (one observation often made by the teachers is that whereas in the original shape the number of circles in a line are two more than the one above, in the shape produced by their rules the number of circles in a line is only one more than the one above.) | 4 |
| 5 | … omit rule 2 | ….. pretends to be the robot and when asked to take lids (rule 1) picks up an even number of lids. The instructor follows all of the rules by making horizontal lines. At some point the rules lead to a horizontal line consisting by two circles. Thus when getting to rule 5 a problematic situation arises since by removing the first and last lid the line is left with zero lids. This makes the teachers to go back to the shape and S4 (Fig. 2) on which they based their rules and observe that all horizontal lines consist by odd number of circles. This leads the teachers to formulate rules for making sure that the number of lids used for the starting line is odd. | 2a-b |
| 6 | … omit rule 1a and 1b | …. pretends to be the robot. When asked at the beginning to take milk bottle lids (rule 1) the instructor picks up only 1 lid. The teachers realize that in this case following the next rules is problematic and thus think about what the robot should do in case by following rule 1the robot takes only one lid. This procedure also makes the teachers test their rules in relation to other numbers (e.g. what will happen if by following rule 1 the robot picks up 2 lids). | 1a-b |

**Table 4** Instructor’s protocol (Act3, Table 1)

| | | |
|---|---|---|
| 1 | Take (as many) lids (as you want). | |
| | a | If you have only taken one lid put it down. You have finished. |
| | b | If you have taken more than one lid continue with the following rule |
| 2 | Put the lids in pairs (two-two). | |
| | a | If there is one left continue with the following instructions |
| | b | If there is no one left take one more and continue with the following instruction |
| 3 | Put the lids one next to each other in a line. This is your starting line | |
| 4 | Take lids and make a new line by putting one lid under each lid of the starting line | |
| 5 | In the new line you made remove the first and last lid. | |
| | a | If there is only one lid left in this new line continue with the next instruction |
| | b | If there is not only one lid left in this new line repeat the last two instructions (instructions 4 and 5) |
| 6 | Take lids and make a new line by putting one lid on top of each lid of the starting line | |
| 7 | In this new line you made remove the first and last lid | |
| | a | If there is only one lid left in this new line continue with the next instruction |
| | b | If there is not only one lid left in this new line continue with the next instruction |
| 8 | Take lids and make a new line by putting one lid on top of each lid of the last line you made | |
| 9 | In this new line remove the first and last lid | |
| | a | If there is only one lid left in this new line you have finished |
| | b | If there is not only one lid left in this new line repeat the last two instructions (instructions 8 and 9) |

**Table 5** Example of rules for programing the robot (Act3, Table 1)

## 4. Discusion: Defining Modeling-based Learning (MbL)

In the previous sections of the paper we saw how through their involvement in the activities the teachers came to better understand a certain structure through their effort to solve problems which involved a process of collecting observation, data and experiences and representing these in different ways. We saw (a) how the teachers used graphical representations and equations to count the circles in the shape (Act 1) which allowed them to 'see' the shape from many different angles and carefully observe the shapes structure, (b) how they used the graphical representations and the equations from Act 1 to hypothesis and thus predict the behaviour of the structure in order to address the problem posed in Act 2 and (c) how they constructed a different type of representation (set of rules) to program a robot to construct shapes like the one studied in the previous activity (Act. 3). It is important to stress out that from this analysis the way MbL was described in the Introduction is supported and in addition it is emphasized that mathematical models 'are powerful tools in predicting the behaviour of complex systems' (Lesh & Harel, 2003).

Modeling though, is often described in the literature as a cyclical procedure involving (a) making systematic observations and/or collecting experiences about the phenomenon under study, (b) constructing a model based on observations and experiences, (c) evaluating the model against, predictive power, and/or explanatory adequacy, and (d) revising the model and applying it in new situations (Louca & Zacharia, 2011). Alternatively, we propose the diagram in fig 2 to describe MbL which better emphasizes the interplay between the different phases which is neither linear nor simply cyclical. The diagram in fig. 2 better describes how learner's in order to address problems observe situation, a structure or a phenomenon and proceed with representing their observations in different ways but at the same time how observations and experiences are collected through the act of representation (constructing a model) and how this reflects on the original problem posed.

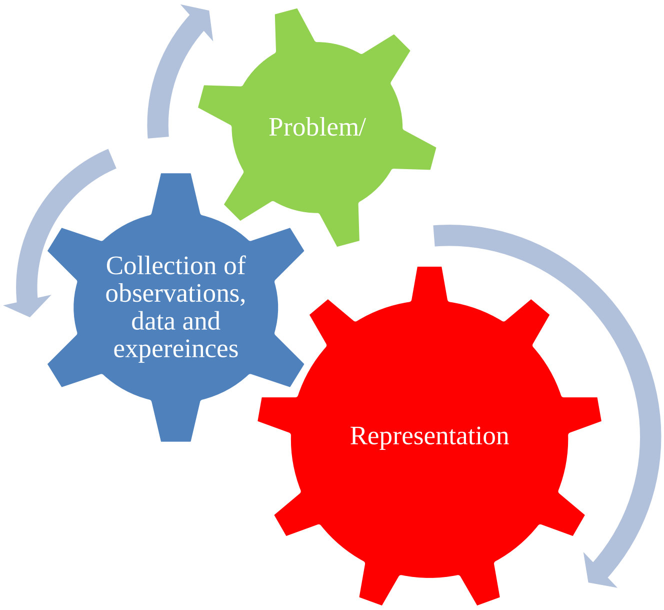



**Figure 3** An overview of Modeling-based Learning

In describing how it feels to be involved in this task sequence (and this applies to the instructor as well as the student/teachers) I would like to use a quote from Ackermann's (2001) attempt to compare Piaget's constructivism with Papert's constructionism: 'Papert's view that diving into unknown situations, at the cost of experiencing a momentary sense of loss, is […] a crucial part of learning. Only when a learner has actually traveled through a world, by adopting different perspectives, or putting on different "glasses," can a dialogue begin between local and initially incompatible experiences.' This paper builds on the conviction that a good way to start educating and supporting teachers to implementing MbL or any other approach that moves away from their traditional practice is by allowing teachers to experience 'diving into unknown situations, at the cost of experiencing a momentary sense of loss' and allowing and supporting them as to use this experience to reflect upon and refine their practice.

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