LEARNING UNIVERSITY MATHEMATICS BY CREATING AND USING FOURTEEN 'MICROWORLDS'

Chantal Buteau[1], Neil Marshall[1], & Eric Muller[1]

Abstract
Since 2001, all mathematics majors and future mathematics teachers at Brock University (Canada) enrol in a sequence of 'microworld-based' courses where they learn to design, program, and use interactive environments for the investigation of a mathematics concept, theorem, conjecture, or real-world situation. By the end of his/her undergraduate studies, each student will have created at least 14 such environments, three on a topic of his/her choosing. In this paper, we describe the overall constructionist learning approach implemented in these courses by exemplifying a student's typical microworld work. This work highlights mathematics content that is overall non-standard in terms of traditional core undergraduate mathematics courses.

Keywords. Microworlds, Sustained Implementation, University Mathematics, Exploratory Objects

1. Introduction

For over a decade, all mathematics majors and future mathematics teachers at Brock University (Canada) have been required, as part of their undergraduate programs, to enrol in a sequence of three-term project-based courses called MICA I-II, an acronym for *Mathematics Integrated with Computers and Applications* [1]. In these courses, students learn to design, program, and use interactive environments, which we have called *Exploratory Objects* (EOs), for the investigation of a mathematics concept, theorem, conjecture, or real-world situation [2]. Every week students meet for a two-hour computer lab and a two-hour lecture where they progressively engage in learning and doing mathematics by creating and using their own EOs [3], which we recently identified as microworld-type environments [4]. The focus in the first-year course is to have students learn about the 'microworld' approach, including learning computer programming, in an easily accessible mathematics context, while the focus evolves in the second-year towards applying the approach with more advanced mathematics [5].

Though there is considerable education research about mathematics microworlds, with some in the university education context (e.g. [6]), there seems to be only a few reports of *sustained* implementation in mathematics classrooms. In fact, Healy and Kynigos [7] stress that mathematics microworlds still have only a "*marginal uptake in schools*" (p.74).

This paper briefly describes the constructionist learning approach implemented in the MICA courses over 10 years by exemplifying the microworld work (14 EOs in total) of a student through her MICA I-II experience. Of the 14 EOs, 11 topics are selected by the instructor, and the

[1] Department of Mathematics and Statistics, Brock University, St.Catharines, Canada.

remaining 3 topics (EO/LO 4,9,14 below) are selected by the students, either individually or in pairs.

2. A Student's Mathematics (Microworld-Type) Exploratory Objects

Ramona Rat enrolled in 2011 in the MICA/Co-op program with a concentration in Statistics. Like a typical student, she completed her Calculus and Linear Algebra requirements in the first semester, and, in the Winter semester, she enrolled in the MICA I course, as well as the second Calculus course. Prior to MICA I, she had had no programming experience like most other MICA I students. In her second-year, she enrolled in second-year mathematics courses, including multi-variable Calculus, Differential Equation, Statistics, Probability, and Linear Algebra courses, as well as in the two-semester MICA II course. Ramona is among our top students with an overall average of 94%.

First-Year Student EO 1. As part of the first MICA I lectures, students are guided to conjecture about prime numbers and Hailstone sequences, while in their lab sessions they learn first basic concepts of VB.Net programming in Visual Studio (i.e., design interfaces, programming variables, loops, and decision structures). The first EO assignment is to select or state a conjecture, and to create and use their EO to explore it. Ramona selected Oppermann's conjecture that states for any integer $n > 1$, there is at least one prime number between $n(n - 1)$ and n^2 and at least another prime number between n^2 and $n(n + 1)$. To see a screenshot of Ramona's EO 1 and all others, see [8]. Using her very first self-created EO, she concluded in her EO written report, that *"[w]hile I was inputting various integers and confirming them, I became more and more surprised that this idea is still a conjecture and not yet a theorem. Even though there is still no proof of it up to this day, my sample data has proven to me that Oppermann's conjecture is a great idea, seeing as how I, personally, believe it works*."

First-Year Student EO 2. Students learn about modular arithmetic, leading to contemporary RSA encryption, while furthering their programming skills and fluency (i.e., functions and modules, and more complex mathematics programs). Their second EO assignment is to implement the RSA encryption algorithm, including encoding, decoding, and randomly generating keys. Ramona, using her EO, encoded and decoded her name, and described the details in her written report: with public key (16171, 19), her encoded name is 3447 – 2372 – 1065 – 1137.

First-Year Student EO 3. The mathematics content moves to discrete dynamical systems. With new programming skills learned (arrays, graphics), students create an EO about the logistic function (one parameter involved), and are guided to systematically explore its behaviour through an interactive lab session. Their third assignment is to create an EO to explore the dynamical system based on a cubic (two parameters involved), and describe its behaviour (Figure 1, left). Ramona (voluntary) mentions in her written report, that "[*b]oth creating and working with this program has assisted me to fully grasp the way a dynamical system works by observing the table, the graphs, and the cobweb with countless test values for a, b, and* x_0."

First-Year Student EO/LO 4 (Original). The MICA I-II courses both culminate at the end of the term in original final projects for which students, individually or in groups of 2 or 3, choose a topic of their own choice; see [9] for examples of student projects. Students are encouraged to select topics that are of high interest to them. For this final project future mathematics teachers can choose to create an environment for the step-wise guided learning of a (school) mathematical concept or skill, that we call *Learning Object* [2]. Ramona and her future teacher partner Carrie, decided to create a LO for Grade 4 students to learn about perimeter. After observing Ramona's friend's Grade

4 younger sister going through the program, they proudly reported that, “*she ended up understanding the concepts reasonably well, which makes our work feel very meaningful*.”

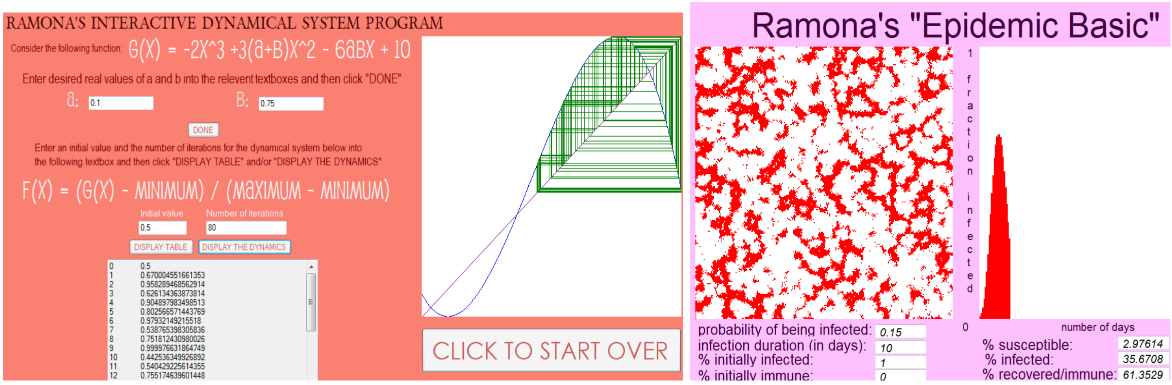

Figure 1. Screen shots of Ramona’s third EO (left) and thirdteen EO (right).

<u>Second-Year Student EO 5.</u> The second-year MICA course starts with the Buffon needle problem that involves dropping a needle onto lined paper and determining the probability that the needle touches a line (and involves Monte Carlo integration). Students also cover the concept of randomness in programming. Their first EO assignment contains four components: i) a modified Buffon needle problem; ii) the area of a stratified region in a rectangle; iii) the area of a unit hyper-sphere, and iv) the Buffon-Laplace problem. Using her EO for iii), Ramona writes that, “*I can confidently conclude that the estimates for the volume converges much quicker to the correct value as n gets large*.” Ramona describes her approach to iv) with many visuals, mentioning “*one must develop the formulae that can be used to calculate values …*”, and after elaborating on the mathematics, cautions that “[*h]owever, this problem isn’t as simple as it looks*” since her initial approach “*cause[d] serious mistakes*“, and finally indicates how she then fixed her model.

<u>Second-Year Student EO 6.</u> Students are introduced to the mathematics of the stock market involving many statistical concepts. Programming-wise, they learn to implement related statistical concepts, explore and experience the Central Limit Theorem, and learn to read data from files. Their assignment contains three components: i) histogram summary of (real data) S&P 500 from 1950 to 2012, and analysis; ii) analysis of a selection of 10 different stocks (of their own choice), and recommendations; and iii) a regression analysis of Walmart stock over a decade. Based on her selection of stocks, Ramona mentions that, she “*would definitely recommend a client to buy the AAPL stock since it’s the most ideal choice when looking at the mean and the average yearly percentage*.” In her analysis of Walmart stock, Ramona indicated that, “*the correlation coefficient is negative... This implies that there is both a very weak correlation (that is, a weak relationship between the current returns and the last returns) and a negative slope (of the regression line). Judging from the “cloud” that the points form, a weak correlation is correct*.”

<u>Second-Year Student EO 7.</u> The course evolves to a different real-world mathematics application: the synchronization of traffic lights. Students are exposed to modelling from the ground up using real data and a simple mathematical concept (time difference). In this ‘Industrial Mathematics in the Community’ EO assignment, students work in groups of 2 to 3 to collect real data at two downtown traffic lights of their choice, and create an EO to analyse and explore a proposed model of synchronization, and to propose their own. Ramona’s team described, “*now that we understand this definition of synchronization, we can come to the conclusion that it is possible to make a clear case that the traffic lights we selected can’t be synchronized simply because…*” They further investigated the model and stated that, “*[o]ur conjecture about the relationship that must hold*

between m_1 and m_2 in order for the two traffic lights Alpha and Beta to be synchronized correctly (that is, that an optimal time occurs in each of the columns) is that $m_1 = m_2$."

Second-Year Student EO 8. The mathematics content covered shifts to Markov Chains which is significantly elaborated in lectures, and involves the concept of equilibrium. In their EO assignment, students apply the mathematics as a model of income demographics and chronic illness. In the former, Ramona found that independently of the proportions in each socio-economic category, "*the equilibrium [of the given model] will always be reached*." She also explored a modified model in which the next generation of high-income wage earners remain so.

Second-Year Student EO/LO 9 (Original). Ramona teamed up with two colleagues and they decided to create an EO to explore the effects of earthquakes (using the Richter scale) on buildings of different ages (i.e. exploring how many of them crash on average). They report, "*[w]e researched some statistics on various websites about the age of buildings that crashed in order to develop the [model]"* and indicate, *"[w]e treat the Richter scale as follows (based on our research)…*". Using their EO, they explored the model they had created, and compared it to known statistics: " *[t]his observation was possible based on both our research and our program.*" They summarized their results on the effects in table form. They added to their initial exploration, "*[t]his general pattern forms several curves which we call equidamage curves. These curves display the most crucial relationship in our study, which is the relationship between the Richter Scale value … and the value … that determines the age distribution of the buildings in the city".*

Second-Year Student EO 10. Students revisit discrete dynamical systems in a more in-depth approach, including an analytical classification of the fixed points of the logistic function. Students create an EO to explore and analyse the bifurcation diagram. Following the assignment guidelines, Ramona very thoroughly analysed the system and reported her findings in her written report using numerous screen shots as support, complemented with Maple algebraic calculations. She observed that, "*there seems to be exactly one downward sloping 'curve' that exists from the third bifurcation point … and on. While some other 'curves' seem to appear and then fade off into the mess, this specific 'curve' seems to be consistently visible."*

Second-Year Student EO 11. Students examine the discrete Lanchester equations that simulate a battle between two armies using an analytical approach, with emphasis on interpretation. For their EO assignment, students simulate battles according to this model, and expand it to include the exploration of random arrival of additional troops (i.e., stochastic processes). Ramona describes, that *"[o]bserving my program …, this battle will end on the 37^{th} day with the X army winning*." Discussing her exploration of the stochastic extension, Ramona brings forth different possible explanations of the model behaviour, and concludes: "*My main finding … was that it is a wasted effort to send too many new troops … Observe the screenshot I have provided … If one let P = 500 instead of P = 2000, approximately the same probability is calculated. It is a waste of funds and equipment to send so many new troops when much fewer can do roughly the same job*."

Second-Year Student EO 12. The mathematics content evolves to continuous dynamical system with a predator-prey biological model, described by the Lotka-Volterra equations, involving Euler's method and phase plane trajectories. Students create an EO to explore balancing this biological system by examining different required case scenarios. Ramona, using her EO, explores, analyses, and interprets the behaviour of the system; for example, she states *"[t]herefore [in this case], since the wolves will have more rabbits to eat due to their population increasing through reproduction, their population is expected to grow as well, which will lead to the rabbit population falling again* *because there are more wolves to eat them. Essentially, the cycle patterns are similar … [as] explained through the comparison of the two screenshots*".

Second-Year Student EO 13. Students are introduced to the theory of cellular automata. The EO assignments culminate, programming-wise, in this challenging EO that involves two-dimensional arrays and the process of cellular automata. Students create simulations of epidemics to examine the effects of inoculation on the spread of epidemics, and their cost. Ramona experienced real-time simulations of required case scenarios (Figure 1, right), and reported her results in her written report in terms of highest susceptible percentage. Building on a simulation of a specific epidemic to estimate its related costs, she found that her "*estimate for the percentage of immunized people that minimizes the cost of the epidemic is 67% by inspection in which the cost is $12,078,150.*"

Second-Year Student EO/LO 14 (Original). For this final original project, Ramona teamed again with her two colleagues. They decided to create an EO to explore the changes in the water supply of Lake Erie (Ontario) over time and explain why and how it changes. Their remarkable real-world EO project involved relatively comprehensive research. Indeed, they indicate that their research was "*a crucial starting point in [their] project, allowing [them] to obtain an understanding for the changes in the water supply of Lake Erie*". They also called private companies to request figures not made public on their websites. In a detailed 26-page report, they thoroughly enumerate and justify, based on their research, all numerical values and assumptions that are used in their (deterministic and stochastic) models and EO, respectively. They were careful that their models mirror reality: "*[i]t was good to have a reference to make sure our calculations, assumptions, etc. were correct*." They analysed six case studies, and added a 'free-will' simulation "*useful for someone who is trying to save Lake Erie*".

3. Discussion

In this paper we briefly described the microworld work (14 EOs in total) of a MICA undergraduate student. The 14 EO projects demonstrate a mathematics content that is overall non-standard in terms of traditional *core* undergraduate mathematics courses. In fact, when the mathematics department designed the core MICA courses, not only was a different pedagogy instituted [3], but also the question of 'what mathematics' was at the heart of the discussion [5]. And this epistemology component is one of the key aspects distinguishing constructionism from constructivism [10]. In particular, it resulted in having students "*do things with a computer that would be impossible without it*" [7, p.65] – see e.g. Ramona's EO 13. Interestingly, the MICA courses were developed independently from constructionism and the microworld literature [5].

In the constructionist approach found in MICA courses, programming is central to the EO activity. When Ramona describes her experience in EO 5, her account suggests inquiry: "*Why doesn't this work right? How can I fix my ideas?*"[11] that she then resolved in a programming context. This aligns well with Wilensky's [6] observation of a university student's microworld experience, where he notes that, "*It was not until [the student] programmed a simulation of the problem that she began to resolve [Bertrand's] paradox*" (p. 272). Interestingly, Ramona indicates that the most challenging EO project was EO 2 due to the programming complexity that she experienced. She further comments that programming was no longer a concern to her afterwards. In our experience, this 'most challenging EO' varies from student to student and this variation points to potential future research, namely: what is it in a student's experience or mathematics maturity that triggers that student's 'overpass' to an almost exclusive focus on mathematics learning in his/her EO work?

One of the key ideas when creating MICA I-II courses was for "*students to be creative and personally involved so that each student could express: 'I did something related to me'* " [5, [p. 4]]. The overall 14 EO activity design can be viewed as shaped around the original three EOs (EOs 4,9,14) on topics selected by the individual students or in pairs. In our experience, students show remarkable ownership, engagement with, and pride in their original EOs [2]. For example, excerpts from Ramona's EO 9 written report suggest that, in the words of Papert [12], "*[t]his [girl] was appropriating mathematics in a deeply personal way*". Not surprisingly, Ramona mentions that her favourite EO was among her original EOs, namely EO 14, *"because [it was] so applicable to a local situation and it was really real… It makes it a real-life feel versus made-up numbers"*, and added: *"I guess it felt like the most important project"*. She indicates that this final EO project is also the EO she is most proud of, since *"it wasn't just programming, it was the most research we had to do for a project. The most work"*. Ramona's reported experience with her EO 14 suggests that it exemplifies the constructionist principle that, "*students need to build models of parts of their world in order to more fully grasp those parts' meaning, substance and dynamics*" [11].

Healy and Kynigos [7] indicate that *"[t]he microworld grows along with the knowledge of its users (Hoyles, 1993)*" (p.65) where 'users' in our paper are exemplified by Ramona as both the creator and user [2]. The authors further indicate that *"[i]t is in the growth that we can locate the essence of constructionism*." [7, p.65]. This sets the direction for future collaborative research grounded on our implementation (e.g., Ramona's cohort produced approximately 560 assigned and 100 original EOs): to carefully examine, in the context of constructionism, how knowledge of individual MICA students, of various achievement levels, evolve alongside their creation and use of EOs.

4. References

[1] B. Ralph, "Mathematics takes an exciting new direction with MICA program," *Brock Teaching,* vol. 1, no. 1, p. 1., 2001. Available: http://www.brocku.ca/webfm_send/18483

[2] E. Muller et al., "Learning mathematics through the design and implementation of Exploratory and Learning Objects," *Int. J. for Technology in Math. Educ.*, vol. 16, no. 2, pp. 63-74, 2009.

[3] C. Buteau and E. Muller, "Teaching roles in a technology intensive core undergraduate mathematics course," in *The mathematics teacher in the digital era: An international perspective on technology focused professional development*, A. Clark-Wilson, O. Robutti, & N. Sinclair Eds. Springer, 2014, pp. 163-185.

[4] N. Marshall and C. Buteau, "Learning by designing and experimenting with interactive, dynamic mathematics Exploratory Objects," *Int. J. for Technology in Math. Educ.*, to be published.

[5] C. Buteau, N. Marshall and E. Muller, "Perception on the Nature of Core University Mathematics Micro-world-Based Courses," in *Proc. of Constructionism 2014*, Vienna, Austria, 2014, to be published.

[6] U. Wilensky, "Paradox, programming, and learning probability: A case study in a connected mathematics framework," *The Journal of Mathematical Behavior*, vol. 4, no. 2, pp. 253-280, 1995.

[7] L. Healy and C. Kynigos, "Charting the microworld territory over time: design and construction in mathematics education," *ZDM,* vol. 42, no. 1, pp. 63-76, 2010.

[8] C. Buteau (2014, May 30), Sequence of 14 MICA Projects [Online]. Available: www.tinyURL.com/micaEOs

[9] Brock University (2014, Apr 13), *Exploratory and Learning Objects created by Brock students in mathematics courses* [Online]. Available: www.brocku.ca/mathematics/studentprojects

[10] R. Noss, "Reconstructing *Constructionism*", in *Proc. of Constructionism 2010*, Paris, France, 2010, pp.1-4.

[11] Constructionism 2014 (2014, May 27), *History of Constructionism* [Online]. Available: http://constructionism2014.ifs.tuwien.ac.at/history

[12] S. Papert and I. Harel, "Situating Constructionism," in *Constructionism*, I. Harel and S. Papert Eds. Ablex Publishing, 1991, pp. 1-11.

[2] Our sincere thanks to Ramona Rat for allowing us to use her MICA work for this paper.