DEVELOPING ELEMENTARY STUDENTS' REASONING OF GEOMETRIC TRANSFORMATIONS THROUGH DYNAMIC ANIMATION

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The aim of this project was to address the need for developing students' reasoning of geometric transformations in elementary school. Our goal was to engage students with transformation-based ideas using the animation software Graphs N' Glyphs. Three teaching experiments were conducted with three pairs of students in Grades 4, 5, and 6. In utilizing Graphs 'N Glyphs around carefully designed tasks, students were able to form generalizations about the way a shape behaves depending on the varied conditions of the transformation performed. Some of these generalizations referred to the use of referent points for performing translations and the role of the center of rotation while performing rotations. These generalizations are seen as foundational for developing the ideas of property, invariance, and congruence in geometry.

Keywords: geometry, transformations, constructionist technologies

1. Developing Students' Transformation-Based Reasoning

The teaching and learning of geometry in schools is principally focused on identifying canonical shapes and matching those shapes to their given names [1, 2]. Learning is constrained to passive observation of static images of shapes on paper, which inevitably limits the engagement and understanding of geometry to holistic representations of shapes. Consequently, misconceptions arise. Examples include young students arguing that a triangle whose base is not horizontal is some other kind of shape, or that a square and a regular diamond are distinct because they are oriented at different angles [3, p. 149]. We begin with the assumption that offering students dynamic experiences of geometry will reduce the incidence of formation of such misconceptions.

This assumption draws on research showing that young children already possess a dynamic spatial sense of shape; they see shapes as malleable and often provide "morphing explanations" [3, p. 142] for shapes they identify as similar. Additionally, research showed that students are able to identify congruency and similarity as early as first grade [4], and also that preschool-age students are able to identify the effects of rigid geometric transformations on isolated figures [5, 6, 7, 8]. These studies suggest that not only do young children have intuitive occurrences of mental transformations of shapes but also have a sense of congruence and similarity [9].

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Geometric transformations (GT) such as translation and rotation can be foundational to the formation of students' geometric conceptions as their dynamic nature provides students with opportunities to connect a greater breadth of geometric concepts. For instance, investigations of invariance, property, and equivalence can be seen to originate with an introduction in GT, where the nature of transformations informs these concepts. Therefore, we support Jones and Mooney's [10] recommendation that students must develop robust understandings of rotations, reflections, translations, similarity and congruency, and be able to connect these transformation concepts, by the end of elementary school. Such investigations of transformations cultivate in students a "geometrical eye," or "the power of seeing geometrical properties detach themselves from a figure" [11, p. 197].

We argue that in order to develop children's transformation-based reasoning, they must be provided with opportunities in which transformations are made salient. While research has suggested that elementary school students are capable of performing as well as identifying the effects of GT [6, 7, 8], the advent of digital technology has opened new opportunities for a more dynamic approach to the study and learning of this concept. We support the argument that by providing students with a medium in which they can explore GT as dynamic processes, they will develop better understandings of the concepts underlying the GT [12]. In other words, following a proposition by Schwartz and Yerushalmy [13] that "important mathematical ideas can be introduced early on in the mathematical education of all students if the introduction is done in the context of interesting and powerful exploratory environments" (p. 7), we advocate for the introduction of the "important ideas" of invariance, symmetry and scale mediated by these environments. Research conducted over the last several decades has confirmed that technology has not only changed the way we teach mathematical concepts but also the nature of knowledge, and thus what is possible to learn [14]. Papert [15] explained:

> I see the computer as helping in two ways. First, the computer allows, or obliges, the child to externalize intuitive expectations. When the intuition is translated into a program it becomes more obtrusive and more accessible for reflection. Second, computational ideas can be taken up as materials for the work of remodeling intuitive knowledge [15, p. 145]

Studies involving the use of computer software for the teaching of geometric topics include the use of LOGO as a way to help develop students' conceptions of angles and angle measurements [16, 17, 18]. Use of the LOGO software also allowed students to examine the dynamic nature of geometric transformation as a means of examining shapes and developing and supporting or refuting conjectures [19, 20]. More recently, Geometer's Sketchpad has been employed as a way to assess high school students' views of the nature of geometric transformations on a plane [21, 22]. Although these studies have utilized geometric software to investigate elements of students' thinking and understanding, they have not focused on the development of students' understanding in elementary school.

Consequently, the aim of this study was to begin to bridge this research gap by exploring the proposition that elementary grade children could learn GT using a multi-representational, dynamic environment called Graphs 'N Glyphs (GnG) [23]. This environment was to provide students with a space to "explore the potentially rich environment of animation and use it as a means for mathematical inquiry" [24, p. 114]. Though the primary original intent of Graphs 'N Glyphs development was to explore a graphical and animation-based environment in which students explored the topics of rational numbers, ratios and proportions, fractions and decimals, and periodic functions, Confrey et al. [25] discovered that the software allowed students to became familiar with

GT by translating, rotating and scaling shapes to complete puzzles. This study extends Confrey and Maloney's work [23] by examining how GnG could facilitate the learning of translation, rotation and scaling by students in elementary school. More specifically, it aims to explore the questions: a) How can GT be introduced to elementary school aged children? and b) To what extent does GnG engage students and enhance student learning of GT?

2. Researching Transformation-Based Reasoning

We designed an exploratory study to investigate how Graphs 'N Glyphs can be used to support elementary students' learning of GT. In particular, we explored the constraints and affordances of Graphs 'N Glyphs (GnG) for thinking-in-change [26] about GT. A series of teaching experiments [27] were conducted, based around the use of specially designed tasks using GnG. The initial interview plan and tasks were informed by the ideas embedded in the study of Confrey, Maloney and colleagues [24]. The participants of the study were three pairs of students, with one pair each drawn from the 4th, 5th, and 6th Grade. All six students attended an elementary school in the eastern United States. The teaching episodes lasted 40 minutes each and occurred over six days for students in 6th grade, and twelve days for students in 4th grade, varying based on the amount of supplemental instruction required and students' rate of progress through the activities.

2.1 The software

GnG provides a means for students to model the motion of objects in two-dimensional space [25]. The students are able to direct GnG to perform specific geometric transformations either singly or as part of a sequence through the use of a timeline or animation sequencer. For example, a student may select "translation" as the desired transformation. The student then enters the desired values for the magnitude of that translation (Figure 1). Enactment of the entered transformation provides feedback to the student, providing the student opportunities to reflect on outcomes and subsequently to construct generalizations of the effects of GT in the plane.

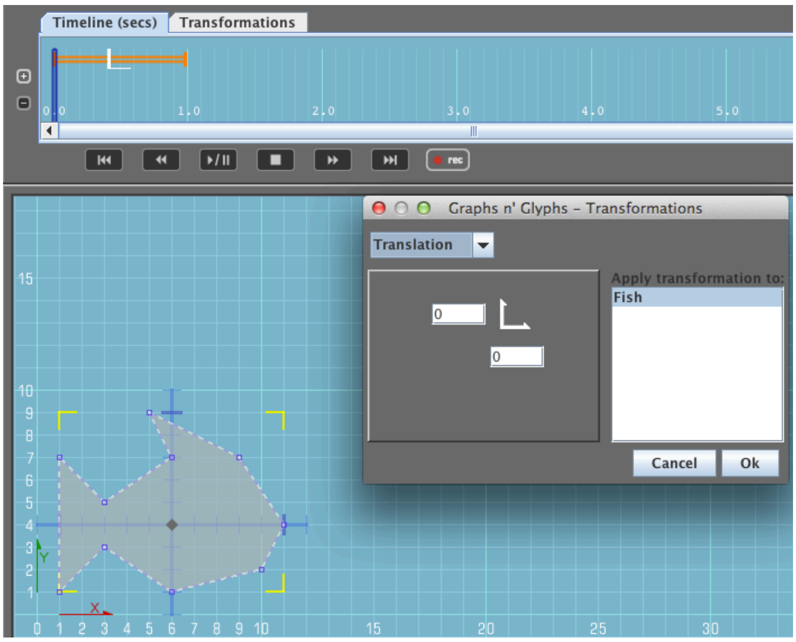

Figure 1: **Using the timeline to perform GT with animation**

2.2 The tasks

Three different types of tasks were used to introduce each of the GTs--translation, rotation and scaling:

a) *Table task:* Shapes (square, rectangle, triangle) were placed on the table, one at a time. Students were asked to close their eyes for 10 seconds. Then they were asked a series of questions such as "What could I have done to the square/rectangle/triangle?" or "What changed and what stayed the same?" The purpose was to examine their initial knowledge of GT and to initiate a discussion focused on transformational invariants.

b) *Coordinate plane task:* The shapes were moved on a coordinate grid and the students were asked the same set of questions. The intent was to help them leverage the additional information they had while working on the grid (namely, the use of coordinates and how these change).

c) *GnG introductory task*: Students were asked to perform a specific GT in GnG using the Transformations dropdown menu (see Figure 1). Students initially explored the way GTs are entered into the GnG Transformation dialog box, using a task calling for predetermined movements of a single shape either in terms of translations or rotations.

d) *GnG matching tasks*: Students were presented two copies of the same figure, A and B, and were asked to apply one or more GTs to figure A so that its location would match that of figure B. These tasks required students to translate, rotate or scale shapes in discrete ways. Students were first asked to record their predictions for the designated transformation before being permitted to use GnG to verify their prediction and modify their conjecture, using the visual feedback offered by the software. Figure 2 shows an example of a matching exercise for translation.

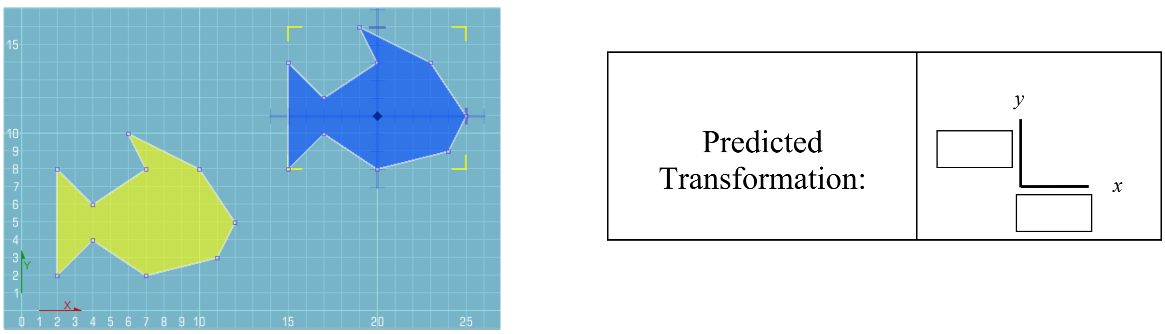

Figure 2: **Use translation to move Shape A to match Shape B**

2.3 Supplementary interventions

In order to ensure that students had sufficient prior knowledge of the coordinate plane and angular measurement to perform the designated GT in GnG, supplemental instruction was provided. Students were presented a series of tasks calling for them to identify given points in the 1st quadrant of the coordinate plane and to determine movements that would be required to move one point to the other. Instruction was confined to the 1st quadrant to avoid confusion with negative coordinate values. Additionally, before discussing rotation, the students' understanding of angle was assessed. While the 5th and 6th graders were able to display appropriate knowledge to solve these tasks, the 4th graders needed to develop their understanding of degree as a measure of angle in order to engage with the tasks. We therefore supported the development of their understanding by first exploring angles as wedges of a circle and then as formed by two rays and an endpoint. Students understanding of an angle was extended to include angles as turns by interacting with GnG.

3. Describing Transformation-Based Reasoning

The focus of analysis was the extent to which the software and the interview tasks reflected our initial conjecture, namely, that GnG could support student learning of transformation-based reasoning. Additionally, we analysed the generalizations students made while working with GnG in terms of 'situated abstractions.' Situated abstractions are generalizations that students form in order to act in specific mathematical contexts, and which are embedded in the particular context where they take place [28]. For example, within GnG, a clockwise rotation requires the use of a negative angle; therefore we explored how students' articulations of transformation-based reasoning were meaningful in relation to the specific features of the GnG environment. The following paragraphs provide examples of the generalizations articulated by the three pairs of students, describing their developing transformation-based reasoning as they interacted with the tasks and the software. Space will not allow for full elaboration of all the GT students engaged with, so the following paragraphs only describe students' experiences with translation and rotation tasks.

3.1 Translations

During the Table and Coordinate plane tasks, students described different ways of translating a shape, such as "up or down, diagonal, or left and right" (Grade 6) and generalized that "translation is when you move a shape but it doesn't change the shape, it just changes its place" (Grade 6), a description that suggests that the student assumes the conservation of shape under translation. All three student-pairs recognized that although the coordinates of points on the figure do change, the shape and size of the figure do not. They also identified the location of the vertices in the initial and final positions of a translation, as well as the nature of the change in location. For instance, the 4th graders determined that "the points are changing to a (7, 5) and a (11, 5)," concluding that "the vertical line [coordinate] never changes. Only the horizontal line [coordinate] changes."

The translation tasks utilizing GnG presented to students were, in order, horizontal translations, then vertical, and finally, diagonal translations (non-zero horizontal and vertical change). Students recognized that when the horizontal change of a translation is positive, the shape moves to the right. Subsequently, when asked to move a shape 8 spaces to the left, the 4th and 6th graders determined that they would have to use an input of "-8." They identified the minus sign as the operation of subtraction, stating that "minus means back or take away" (Grade 4) and "subtract 8 places" (Grade 6). In contrast, the 5th graders described their understanding of a minus sign as "negative is the opposite of positive" (Grade 5). Eventually, each of the groups was also able to move shapes upward (+) and downward (-), thus performing vertical translations, and experiencing the minus sign (4th and 6th grade) and the negative sign (5th grade) as a direction that is the reverse of the positive (right or upward) direction. Additionally, although the worksheets provided two boxes for assigning predicted translation inputs to allow diagonal translations (one input for the horizontal change and one for the vertical change), each group applied both vertical and horizontal translations automatically as a single input, neglecting an input in the second prediction box.

In attempting to match the shapes, students used referents (corresponding points on each shape), to determine the nature of the translation needed in the mapping. For example, while trying to match star A to star B in Figure 3, 6th grade students argued: "I chose the top point for A and counted how many points to go up first… 6, and then in order for it to move left to reach this point it would be 6 places." By being able to experiment with different values and using feedback from the resulting translations, students noticed that even if they chose different referents, the translation would remain unchanged. Therefore, they recognized that they only had to determine the distance between any pair of corresponding points, as is seen by the circled upper vertex (Figure 3). This suggests that after working with tasks using GnG, the students were able to construct and perform mappings to translate figures. They were also able to attend to the magnitude of these translations, which would have gone unexamined had they only used motions and surfaces that did not require attention to position information relative to a background grid.

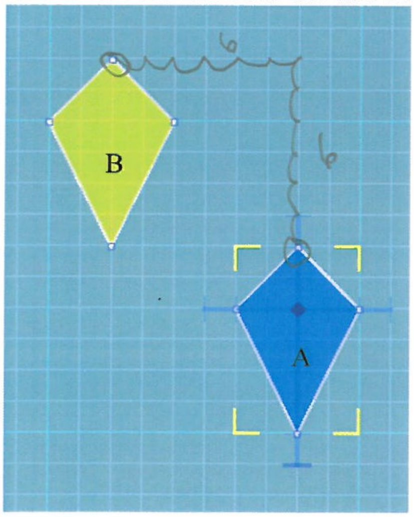

Figure 3: **A drawing of one of the sixth graders work to determine a translation**

3.2 Rotations

During the Table task, students identified several key features of rotation, such as "[they can be] clockwise and anti-clockwise." (Grade 4); "its place changes, but not the size"(Grade 5); and "[the shape] changes where it faces. Its shape would be the same" (Grade 4) Students also recognized the properties of shape under rotation. For instance, the 6th graders discovered that the result of the rotation of a square is harder to determine than that of a triangle or rectangle, because it has "the same kind of length and angles." During the coordinate plane task, students extended this generalization by referring to the change of coordinates after a full rotation depending on the location of the center of rotation relative to the center of the figure. (GnG is designed to permit students to change the center of rotation, point of invariance, and line of symmetry for rotations, scaling, and reflections, respectively.) For example, "If you do [rotate] it [the square] from the right angle [corner, or vertex], it would be in different coordinates [as it rotates]. But if you do [rotate] it [the square] like this [using the center of the square as the center of rotation], it would be the same coordinates" (Grade 5). Thus they demonstrated that rotating a square around one of its vertices generates a large circle with the square rotation outside of its original boundaries, while rotating a square around its center, brings the square back inside its original boundaries, several times during a full rotation.

When working with the GnG rotational matching tasks, in order to achieve the desired rotation, the students needed to ensure that the center of rotation was correctly positioned, and to determine a direction and magnitude for the desired rotation. The students first explored rotation in GnG through a series of prompts asking them to perform rotations around centers of rotation located at various points on the figure. GnG is designed so that the angle values of counter-clockwise rotations are positive and the angle values of clockwise rotations are negative. In the context of the software, each group of students extended their generalizations about the role of the negative sign as reversing the (normal, or clockwise) direction of rotation. They explained that they used the minus sign "because the software goes like that (gesturing anti-clockwise), so if you want to do it like that (gesturing clockwise)… you put the negative sign" (Grade 5).

Students also noticed that in a rotation, the path travelled by a shape around the center of rotation "goes exactly as a circle… It is like an invisible circle is surrounding this" (Grade 4). Additionally, when asked to form conjectures for performing both clockwise and anti-clockwise rotations for each matching task, the 6th graders used the knowledge that a complete circle is 360 degrees to determine the magnitude of the clockwise rotation when they knew the counter-clockwise one, explaining for example that: "We can do 360 minus 165" (Grade 6).

Students recognized that the center of rotation was a position that can be assigned according to their needs in a particular task. The recognized that some of the more advanced matching tasks required that they change the position of the "pivot" (the name of the center of rotation in GnG), and also that it was possible to place the pivot 'outside' the shape being transformed. For example, in the task presented in Figure 4, each group recognized that one strategy for placing shape A directly onto shape B was to place the center of rotation for shape A in the "center" of the space between the two stars.

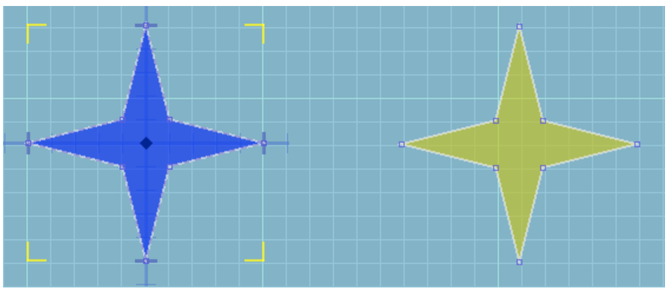

Figure 4: **A rotation task for which the center of rotation can be located midway between the shapes.**

As they solved the transformation tasks, students generalized about invariance of the center of rotation (for a given rotation), as well as about the distance between the shape and its assigned pivot. The students in 4th grade noticed, "Where you put it [the center of rotation], it makes it stick right there, so when you move it [the shape], it [the shape] rotates around the pivot." Students in 5th grade discovered the pivot's role in constraining a shape's motion to a circular path during rotation: "It doesn't let it [the shape] move anywhere other [than] where it would move if you just turn it in circles." The students formed generalizations about the magnitude and direction of rotations as well as about the role of the center of rotation.

4. Discussion and Implications

Working the tasks in the GnG microworld [29], students predicted the types, magnitudes, and directions of transformations necessary to solve the tasks. They analyzed and refined those predictions in light of feedback provided by the animation software. Students were able to form generalizations about the nature of transformations. Conceptually, these generalizations may be seen as foundational for investigations of invariance and congruency. Pedagogically, they may also provide an expanded vision of what elementary-age students are capable of learning about GT when tasks and software provide them with opportunities to do so. The generalizations students formed can be regarded as a basis for subsequent, more sophisticated, understandings of GT. In particular, the work reported here suggests that by providing students with opportunities to investigate relationships between transformations and the geometric space in which those transformations are enacted, GnG has the potential to dynamically support the advancement of students' thinking of GT.

We aim to conduct more teaching experiments to develop "superseding models" [30] of students' thinking. We are also interested in identifying the mechanisms that would support efforts to scale transformation-based learning experiences using GnG to classroom situations. This would including developing a next version of the software for increased usability by elementary school students, such as redesigning the timeline tools and enriching the tools for creating new shapes. Due to students' exposure to internet-base video and audio technology, it is a particular epistemological challenge to provide software environments that require students to think through the geometric and mathematical basis of movement and transformations, instead of simply dragging and dropping with automated tools. GnG offers a design that supports their exploration of the underlying mathematics of movement in 2D space.

Overall, this study showed that by exploring transformations early on, elementary students can be introduced to less formal but nonetheless foundational versions of the concepts of congruence and similarity as well as the idea of invariance. They can also be better prepared to develop transformation-based reasoning, including making connections to related concepts of geometry, like properties of shapes, mapping, magnitude, direction, ratio, and negative numbers. The aim is to offer 'gears'-like [15] experiences to elementary-age students that will prepare them to acquire deeper understandings of advanced mathematical ideas in secondary and higher education.

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